Discrete & Computational Geometry manuscript No.
(will be inserted by the editor)

Uniform edge-c-colorings of the Archimedean Tilings

Laura Asaro · John Hyde · Melanie Jensen · Casey Mann · Tyler Schroeder

Received: date / Accepted: date

Abstract In the book *Tilings and Patterns* by B. Grunbaum and G. S. Shephard, the problem of classifying the uniform edge-c-colorings of Archimedean tilings of the Euclidean plane is posed. This article provides such a classification.

Keywords tilings · uniformity · edge-coloring

PACS 05B45 · 52C20 · 54E15 · 05C15

1 Preliminaries

A *plane tiling* $\mathscr{T}$ is a countable family of closed topological disks $\mathscr{T} = \{T_1, T_2, ...\}$ that cover the Euclidean plane $\mathbb{E}^2$ without gaps or overlaps; that is, $\mathscr{T}$ satisfies

This work was supported by NSF grant DMS-1062740.

Laura Asaro
East Central University
E-mail: laugasa@email.ecok.edu

John Hyde
The University of Texas at Tyler
E-mail: jhyde2@patriots.uttyler.edu

Melanie Jensen
Tulane University
E-mail: mjensen1@tulane.edu

Casey Mann
University of Washington Bothell
E-mail: cmann@uwb.edu

Tyler Schroeder
North Central College
E-mail: tyler-schroeder@uiowa.edu

1. $\bigcup_{i\in\mathbb{N}} T_i = \mathbb{E}^2$, and
2. $\text{int}(T_i) \cap \text{int}(T_j) = \emptyset$ when $i \neq j$.

The T_i are called the *tiles* of $\mathscr{T}$. The intersection of any two distinct tiles can be a set of isolated arcs and points. These isolated points are called the *vertices* of the tiling, and the arcs are called the *edges* of the tiling. In this paper, only tilings whose tiles are regular polygons are considered. The straight segments comprising the boundary of a polygon will be called *sides* and the endpoints of these straight segments will be called *corners*. If the corners and sides of the polygons in a tiling coincide with the vertices and edges of the tiling, then the tiling is said to be *edge-to-edge*.

In an edge-to-edge tiling in which every tile is a regular polygon, a vertex v is of *vertex type* $a_1, a_2, \ldots, a_n$ if an a_1-gon, an a_2-gon, …, and an a_n-gon meet at v in that order (any cyclic permutation or reverse ordering is equivalent). Any tiling in which every vertex is of type $a_1, a_2, \ldots, a_n$ is said to be of *Archimedean type* $(a_1, a_2, \ldots, a_n)$. It is well known that there exist precisely 11 distinct edge-to-edge tilings by regular polygons such that all vertices are of the same type, and these are (3^6), $(3^4.6)$, $(3^3.4^2)$, $(3^2.4.3.4)$, $(3.4.6.4)$, $(3.6.3.6)$, (3.12^2), (4^4), $(4.6.12)$, (4.8^2), and (6^3) (Fig. 1, [1]). These tilings are called the *Archimedean tilings* or *uniform tilings*. While these two terms describe the same 11 tilings, the terms "Archimedean" and "uniform" confer two different meanings. Archimedean refers to edge-to-edge tilings by regular polygons that are *monogonal* (that is, a tiling in which every vertex, together with its incident edges, forms a figure congruent to that of any other vertex and its incident edges). On the other hand, uniform refers to edge-to-edge tilings by regular polygons that are *isogonal* (that is, a tiling in which the vertices of the tiling are all in the same transitivity class with respect to the symmetry group of the tiling). It is coincidence that these two notions produce the same 11 tilings. However, if the ideas of uniform and Archimedean are generalized in the natural way to *k-uniform* and *k-Archimedean*, it is known that k-uniform is more restrictive than k-Archimedean when $k \geq 2$ [1]. Two vertices that are in the same transitivity class with respect to the symmetry group of the tiling will be called *equivalent* vertices. So, in a uniform tiling, all vertices are equivalent.

Let $\mathscr{T}$ be a uniform tiling. If each edge of $\mathscr{T}$ is assigned one of c colors so that each of the c colors is represented in the tiling, the resulting edge-colored tiling $\mathscr{T}_c$ is an *edge-c-coloring* of $\mathscr{T}$. $\mathscr{T}_c$ is *Archimedean* if every vertex of $\mathscr{T}_c$, together with its incident edges, forms a figure that can be mapped by a color-preserving isometry to any other vertex in $\mathscr{T}_c$ and its incident edges. $\mathscr{T}_c$ is *uniform* if for any two vertices in $\mathscr{T}_c$, there is a color-preserving symmetry of $\mathscr{T}_c$ that maps the first vertex onto the second. In the case of edge-c-colored tilings, it will be seen that there are only finitely many uniform edge-c-colorings, but there are infinitely many distinct Archimedean edge-c-colorings.

Let $\mathscr{T}_c$ be a uniform edge-c-coloring in which the underlying uncolored uniform tiling is of type $(a_1.a_2.\ldots.a_n)$. If for an arbitrary vertex V of $\mathscr{T}_c$ the color of the edge meeting V between polygon a_i and polygon a_{i+1} is denoted

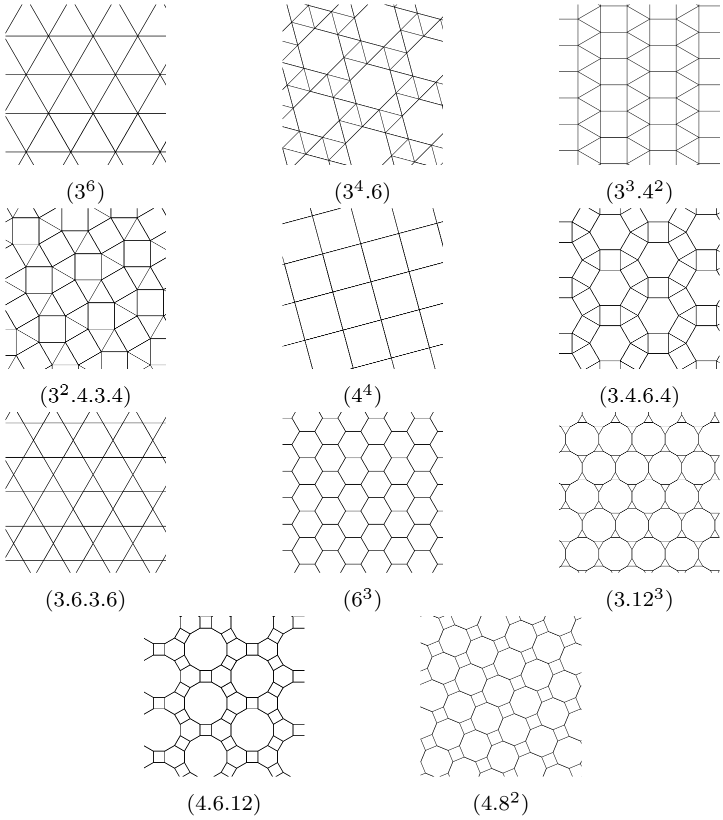


Fig. 1: The 11 uniform tilings.

by c_i, then we will say V has *vertex color configuration* $a_{1_{c_1}} a_{2_{c_2}} \cdots a_{n_{c_n}}$. Since $\mathscr{T}_c$ is uniform, every vertex will have the same vertex color configuration, so we will say $\mathscr{T}_c$ is of type $(a_{1_{c_1}} a_{2_{c_2}} \cdots a_{n_{c_n}})$.

Two vertex color configurations are considered to be *equivalent* if one can be obtained from the other via a combination of cyclic permutations, reverse orderings (including the placement of subscripts), and trivial label renaming. For example, in Fig. 2, the edges incident to a vertex of type $3^3.4^2$ have been assigned colors a, b, c, and d. Starting with the edge labeled a and working clockwise, the vertex has color configuration $3_a 3_b 3_c 4_d 4_c$. Starting from the edge labeled b and working counterclockwise yields the equivalent vertex color configuration $3_b 3_a 3_c 4_d 4_c$; this equivalence can be seen as first cyclically permuting and then reversing the order of this vertex color configuration:

$$3_a 3_b 3_c 4_d 4_c = 3_c 4_d 4_c 3_a 3_b = 3_b 3_a 3_c 4_d 4_c$$

(the colored text is intended to help reader keep track of the changes). Also, by interchanging the labels c and d in the previous vertex color configuration, we obtain the equivalent vertex color configuration $3_b 3_a 3_d 4_c 4_d$.

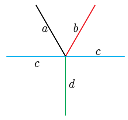


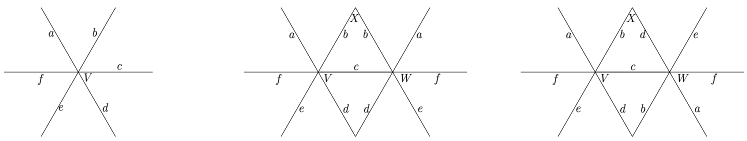
Fig. 2: Vertex color configuration $3_a3_b3_c4_d4_c = 3_b3_a3_c4_d4_c = 3_b3_a3_d4_c4_d$

2 Main Results

Theorem 1 *There are a total of 109 uniform edge-c-colorings of the uniform tilings.*

In Table 13 the vertex color configurations admitting uniform edge-c-colorings and the corresponding number of edge-c-colorings admitted are given. Figures for each of the 109 colorings are presented in Section 5.1.

Clearly the maximum possible number of colors for any uniform edge-c-coloring is $c = 6$, and the only uniform tiling that could possibly have a vertex with incident edges of 6 different colors is (3^6). But, it is quickly seen that no edge-6-colorings of (3^6) exists. Fig. 3a shows a vertex V of type 3^6 whose incident edges have 6 colors, and Figures 3b and 3c show that if any vertex W sharing a common edge with V is colored in the same way, a second vertex X sharing a common edge with V will be forced to have its incident edges colored in a way not consistent with the coloring indicated in Fig. 3a. Therefore only values of c satisfying $1 \leq c \leq 5$ need to be considered.



(a) (b) (c)

Fig. 3: No uniform (or Archimedean) edge-6-colorings of (3^6) exist.

There are two steps in finding all edge-c-colorings of the uniform tilings

1. For each of the 11 vertex types of the uniform tilings and for each value of c, $1 \leq c \leq 5$, determine the complete list of possible vertex color configurations.
2. For each vertex color configuration, determine all symmetrically distinct uniform edge-c-colorings admitted.

Step 1 of that process is straight forward and will be illustrated by example in Section 3. Step 2 is complicated somewhat by the fact that some vertex color configurations admit an uncountable number of nonuniform Archimedean edge-c-colorings. For example, the vertex color configuration in Fig. 4 admits uncountably many nonuniform Archimedean edge-3-colorings of (4^4).

In order to overcome this difficulty, it is first observed that a uniform edge-c-coloring is periodic. A tiling is *periodic* if its symmetry group contains two translations in nonparallel directions. Similarly, a uniform edge-coloring is *color-periodic* if its symmetry group contains at least two color-preserving translations in nonparallel directions. While the (uncolored) uniform tilings are periodic and the symmetry group of any uniform edge-coloring overlaying a uniform tiling is a subgroup of the symmetry group of the underyling uncolored uniform tiling, it is not obvious that a uniform edge-coloring of a uniform tiling is periodic. Indeed, there are many nonperiodic Archimedean edge-colorings. For example, Fig. 4 shows an Archimedean but nonuniform edge-3-coloring of (4^4) that is nonperiodic with respect to color preserving symmetries. Conveniently, uniform edge-c-colorings are isogonal, and isogonal tilings have been classfied into 93 types, all of which are periodic [1]. *Hence, a uniform edge-c-coloring is color-periodic.* It is further established in Lemma 1 that a “small” *period parallelogram* (i.e. a patch of tiles in the tiling that tiles the plane by translations) can be found for any uniform edge-c-coloring, *ensuring that there are only finitely many uniform edge-c-colorings.*

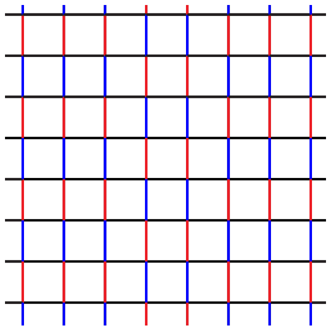


Fig. 4: An example of a nonuniform Archimedean edge-coloring of (4^4) that is not color-periodic.

Before stating Lemma 1, a few definitions are necessary. A vector $\boldsymbol{\tau}$ of minimal magnitude that specifies a translational symmetry of a uniform (uncolored) tiling $\mathscr{T}$ is a called *minimal translation* of $\mathscr{T}$. Two vertices in the tiling that are equivalent by a minimal translation (or its inverse) will be called *translationally adjacent.* If V is a vertex of $\mathscr{T}$, then $\boldsymbol{\tau}(V)$ will be called its *successive vertex in the dirction of* $\boldsymbol{\tau}$ and $-\boldsymbol{\tau}(V)$ will be called its *preceding vertex in the dirction of* $\boldsymbol{\tau}$. Let $\mathscr{T}_c$ be a uniform edge-c-coloring of $\mathscr{T}$ and let V_0 be any vertex of $\mathscr{T}_c$. The vertices of $\mathscr{T}_c$ that are color-preserving translates of V_0 are said to be in the same *aspect* as V_0. Note that two vertices that are in the same aspect in $\mathscr{T}_c$ are not necessarily equivalent in $\mathscr{T}_c$. Because the uniform tiling underlying $\mathscr{T}_c$ is periodic, there is a bi-infinite sequence of vertices in $\mathscr{T}$, $\ldots V_{-2}, V_{-1}, V_0, V_1, V_2, \ldots$ such that V_i is translationally adjacent to V_{i-1} and V_{i+1}. While the V_i are translates of one another in the underlying uniform tiling, in the uniform edge-c-coloring $\mathscr{T}_c$, the V_i may appear in several possible aspects, so there is a corresponding sequence of aspects, $\ldots \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \ldots$. Again, it is emphasized that even when two vertices in the sequence V_i are in the same aspect, the symmetry that takes one to the other may be a translation, a rotation, a reflection, or a glide-reflection.

Lemma 1 *If $\boldsymbol{\tau}$ is a minimal translation in a uniform tiling $\mathscr{T}$, then any uniform edge-c-coloring $\mathscr{T}_c$ of $\mathscr{T}$ admits a translation of magnitude $m \leq 6\,\|\boldsymbol{\tau}\|$. Moreover, $\mathscr{T}_c$ is periodic and has a period parallelogram of dimensions at most $6\,\|\boldsymbol{\tau_1}\| \times 6\,\|\boldsymbol{\tau_2}\|$ where $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$ are minimal translations of $\mathscr{T}$.*

Proof Let $\mathscr{T}_c$ be a uniform edge-c-coloring of $\mathscr{T}$. Let $V_0, V_1, \ldots, V_4$ be consecutive translationally adjacent vertices of $\mathscr{T}$ with minimal translation $\boldsymbol{\tau}$ and let γ_i be the color preserving symmetry of $\mathscr{T}_c$ that maps V_i to V_{i+1}. If γ_i is $\boldsymbol{\tau}$, the result is immediate, so suppose none of the γ_i is $\boldsymbol{\tau}$. Also note that if γ_i is a glide reflection parallel to $\boldsymbol{\tau}$, then γ_i^2 is a translation parallel to $\boldsymbol{\tau}$ with magnitude $2\,\|\boldsymbol{\tau}\|$. So also suppose that none of the γ_i is a glide reflection parallel to $\boldsymbol{\tau}$.

The proof will proceed by cases based the types of the inner two isometries, γ_1 and γ_2.

- Case 1: γ_1 *and* γ_2 *are both glide reflections.* If γ_1 and γ_2 are parallel glide reflections, the collinearity of V_0, V_1, and V_2 and a routine computation reveal that $\gamma_2 \circ \gamma_1$ is a translation parallel to $\boldsymbol{\tau}$ with magnitude $2\,\|\boldsymbol{\tau}\|$. If, γ_1 and γ_2 are nonparallel glide reflections, then γ_1^2 and γ_2^2 are nonparallel translations of $\mathscr{T}_c$ of maginude $m \leq 4\,\|\boldsymbol{\tau}\|$, and the period parallelogram spanned by these translations has dimensions $4\,\|\boldsymbol{\tau}\| \times 4\,\|\boldsymbol{\tau}\|$.
- Case 2: γ_1 *is a reflection and* γ_2 *is a glide reflection.* Then there must be a color preserving glide reflection from V_1 to V_0, and so this case reduces to Case 1.
- Case 3: γ_1 *and* γ_2 *are reflections.* In this case, $\gamma_2 \circ \gamma_1$ is a translation parallel to $\boldsymbol{\tau}$ and of magnitude $2\,\|\boldsymbol{\tau}\|$.
- Case 4: γ_1 *is a rotation and* γ_2 *is a glide reflection.* First, note that if the rotation is not 180°, then there exist two nonparallel glide reflections, γ_2 and $\gamma_2 \circ \gamma_1$, and this reduces to Case 1. If, however, the rotation is 180°, then consider the γ_3. If this is a reflection, then it reduces to Case 3. If it is a glide reflection, then this reduces to Case 1. If on the other hand it is a rotation, then it must be 180° since it would otherwise reduce to Case 1. Then the composition $\gamma_3 \circ \gamma_1$ is a translation parallel to $\boldsymbol{\tau}$ of magnitude $4\,\|\boldsymbol{\tau}\|$.
- Case 5: γ_1 *is a rotation and* γ_2 *is a reflection.* First note that if the rotation is 180°, then the composition $\gamma_2 \circ \gamma_1$ is a glide reflection defined by $2\boldsymbol{\tau}$, and thus there is a translation defined by $4\boldsymbol{\tau}$. If γ_1 is not a 180° rotation, then $\gamma_2 \circ \gamma_1$ and $\gamma_2 \circ \gamma_1^2$ are nonparallel glide reflections and this case reduces to Case 1.
- Case 6: γ_1 *and* γ_2 *are both rotations.* It may be assumed that all symmetries along the line in question are rotations, for if they were not, the problem

would reduce to a previous case. The only rotational symmetries of the unform tilings are of angles that are multiples of 60° and 90°. From this it is seen that every sequence of 6 rotations has a subsequence whose angles sum to a multiple of 360°; first note that each proper rotation has, from among the possible rotational angles, a positive additive inverse modulo 360° (e.g. the positive additive inverse of 120° is 240°). In enumerating all length-6 sequences of rotations that do not contain a subsequence whose angles sum to a multiple of 360°, it is seen that that there are at most 5 choices for the first rotation, and because each rotation has a positive inverse modulo 360°, there are at most 4 choices for the second rotation. For the third rotation, there are at most 3 choices left since the third rotation cannot be the positive additive inverse modulo 360° of the second rotation and the sum of the first two rotations. Similarly, there are at most 2 choices left for the fourth rotation, and 1 for the fifth. And so it is seen that every length-6 sequence of rotations has a subsequence whose angles sum to a multiple of 360°. Since the composition of rotations is a translation if the angles sum to a multiple of 360°, there is a translation in the direction of $\boldsymbol{\tau}$ whose magnitude is $m \leq 6\,\|\boldsymbol{\tau}\|$.

Since $\mathscr{T}$ is periodic, there is a second minimal translation $\boldsymbol{\sigma}$ that is not parallel to $\boldsymbol{\tau}$. In each case above, either a translation of $\mathscr{T}_c$ in the direction of τ of no more than $6\,\|\boldsymbol{\tau}\|$ was identified or a pair of nonparallel translations of magnitude less than $4\,\|\boldsymbol{\tau}\|$ was identified, and the same arguments may be applied to a line of consecutive vertices in the direction of $\boldsymbol{\sigma}$. Thus, a period parallelogram of $\mathscr{T}_c$ of dimensions no more than $6\,\|\boldsymbol{\tau}\| \times 6\,\|\boldsymbol{\sigma}\|$ exists.

Next, we provide a local condition for determining if a given Archimedean edge-c-coloring is uniform.

Lemma 2 *Let $\mathscr{T}_c$ be an Archimedean edge-c-coloring of $\mathscr{T}$ and let V be a vertex in $\mathscr{T}_c$. If V can be mapped to each adjacent vertex by a color-preserving symmetry of $\mathscr{T}_c$, then $\mathscr{T}_c$ is uniform.*

Proof Let W be any vertex of $\mathscr{T}_c$. Let γ be a edge path in $\mathscr{T}_c$ from V to W. Then there is a finite sequence of vertices $V, V_1, V_2, \ldots, V_n, W$ on γ such that consecutive vertices in this sequence are endpoints of the same edge. By hypothesis there is a symmetry α_1 of $\mathscr{T}_c$ taking V to V_1. The same symmetries that take V to its adjacents also take V_1 to its adjacents, and in particular, there is a symmetry of $\mathscr{T}_c$, α_2, taking V_1 to V_2. Continuing in this way, it is seen that there is a sequence of symmetries $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ whose composition takes V to W.

3 Illustrating the Enumeration Process

3.1 Enumerating the Vertex Color Configurations

For each vertex type, there is a small finite number of vertex color configurations. The process of enumerating the vertex color configurations will be illustrated for vertex types 4.6.12 and 3^6, and the other vertex types are handled in exactly the same way, and the results are given in Tables 1 - 11. For both cases, let C be the number of colors under consideration. The colors will be called a, b, c, d, and e, and $|a|$, $|b|$, $|c|$, $|d|$, and $|e|$ will denote the number of edges colored a, b, c, d, and e, respectively.

3.1.1 Enumerating Vertex Color Configurations of Type 4.6.12

If $C = 1$, the only vertex color configuration (up to trivial equivalence) is $4_a6_a12_a$.

If $C = 2$, then there are two cases to consider: $|a| = 2$ and $|b| = 1$, or $|a| = 1$ and $|b| = 2$. However, since trivially interchanging the colors a and b yields equivalent vertex color configurations, only the case $|a| = 2$ and $|b| = 1$ needs to be considered. Thus, the placement of the b determines the vertex color configuration, and so there are three possibilities: $4_a6_a12_b$, $4_a6_b12_a$ and $4_b6_a12_a$. In the interest of listing our possible vertex color configurations in lexicographical order in Table 10, the vertex color configuration $4_b6_a12_a$ has been converted to the equivalent $4_a6_b12_b$. These three possibilities are clearly distinct since, for instance, an edge between a square and a hexagon is in a different edge-transitivity class in the underlying uniform tiling than an edge between a hexagon and a 12-gon. Contrast this with the vertex color configuration $6_a6_a6_b$ of type 6^3; In this case, $6_a6_a6_b = 6_a6_b6_a = 6_b6_a6_a = 6_a6_b6_b$ is the only vertex color configuration with $c = 2$ for vertex type 6^3, owing to the fact that all three edges are in the same transitivity class in the underlying uniform tiling.

If $C = 3$, the only vertex color configuration is $4_a6_b12_c$.

3.1.2 Enumerating Vertex Color Configurations of Type 3^6

If $C = 1$, the only vertex color configuration is $3_a3_a3_a3_a3_a3_a$.

If $C = 2$, let the two colors be a and b. Because the colors a and b can be interchanged to generate trivially equivalent vertex color configurations, we may assume $|b| \leq 3$. If $|a| = 5$ and $|b| = 1$, the placement of the b determines the vertex color configuration, and since in the underlying uniform tiling all edges are equivalent, there is only one vertex color configuration in this case; namely $3_a3_a3_a3_a3_a3_b$. If $|a| = 4$ and $|b| = 2$, then the angle between the two edges labeled b may be 60°, 120°, or 180°, yielding vertex color configurations $3_a3_a3_a3_a3_b3_b$, $3_a3_a3_a3_b3_a3_b$, and $3_a3_a3_b3_a3_a3_b$. Lastly, if $|a| = 3$ and $|b| = 3$, notice that there are two vertex configuration in which two edges labeled a are adjacent, $3_a3_a3_a3_b3_b3_b$ and $3_a3_a3_b3_a3_b3_b$, and if no two edges labeled a are adjacent the only vertex configuration is $3_a3_b3_a3_b3_a3_b$. Thus, when $c = 2$, there are a total of 7 vertex color configurations for 3^6.

If $C = 3$, let the three colors be a, b, and c. Without loss of generality, assume that $|c| \leq |b| \leq |a|$. Thus, there are three possibilities:

- $|a| = 4$, $|b| = 1$, $|c| = 1$

– $|a| = 3$, $|b| = 2$, $|c| = 1$
– $|a| = 2$, $|b| = 2$, $|c| = 2$.

In the case that $|a| = 4$, the angles between the edges labeled b and c can be 60°, 120°, or 180°, yielding $3_a3_a3_a3_a3_b3_c$, $3_a3_a3_a3_b3_a3_c$, and $3_a3_a3_b3_a3_a3_c$. In the case that $|a| = 3$, if all three of the edges labeled a are consecutive, there are two possible ways to arrange the remaining edges, giving $3_a3_a3_a3_b3_b3_c$ and $3_a3_a3_a3_b3_c3_b$. If only two edges labeled a are adjacent, the possible configurations are $3_a3_a3_b3_a3_b3_c$, $3_a3_a3_b3_a3_c3_b$, and $3_a3_a3_c3_a3_b3_b$. In Table 13, $3_a3_a3_c3_a3_b3_b$ is changed to $3_a3_a3_b3_a3_c3_c$ to accomodate listing the vertex configurations in lexicographical order. If none of the 3 edges labeled a are adjacent, the only possible configuration is $3_a3_b3_a3_b3_a3_c$. Finally, in the case that $|a| = 2$, the vertex configurations in which the two edges labeled a are adjacent are $3_a3_a3_b3_b3_c3_c$, $3_a3_a3_b3_c3_b3_c$, and $3_a3_a3_b3_c3_c3_b$. If the angle between the two edges labeled a is 120°, the corresponding vertex configuration is $3_a3_b3_a3_c3_b3_c$. If the angle between the two edges labeled a is 180°, then the one vertex configuration is $3_a3_b3_c3_a3_b3_c$.

If $C = 4$, the edges of the vertex color configurations will be marked a, b, c, and d. Assuming $|d| \leq |c| \leq |b| \leq |a|$, there are two cases:

– $|a| = 3$, $|b| = |c| = |d| = 1$
– $|a| = 2$, $|b| = 2$, $|c| = |d| = 1$

In the case where $|a| = 3$, proceed as before and consider the possibilities that the three edges labeled a are either consecutive, two are adjacent, or no two are adjacent. This gives $3_a3_a3_a3_b3_c3_d$, $3_a3_a3_b3_a3_c3_d$, and $3_a3_b3_a3_c3_a3_d$. In the case where $|a| = 2$, there are the three possibilities that the angle between the two edges labeled a is 60°, 120°, or 180°, and these three possibilities give the following vertex color configurations.

$3_a3_a3_b3_b3_c3_d$
$3_a3_a3_b3_c3_b3_d$
$3_a3_a3_c3_b3_b3_d = 3_a3_a3_b3_c3_c3_d$
$3_a3_a3_b3_c3_d3_b$
$3_a3_b3_a3_b3_c3_d$
$3_a3_b3_a3_c3_b3_d$
$3_a3_c3_a3_b3_d3_b = 3_a3_b3_a3_c3_d3_c$
$3_a3_b3_c3_a3_b3_d$

If $C = 5$, with edge colors a, b, c, d, and e, it may be assumed that $|a| = 2$ and $|b| = |c| = |d| = |e| = 1$. There are three possible vertex configurations: $3_a3_a3_b3_c3_d3_e$, $3_a3_b3_a3_c3_d3_e$, and $3_a3_b3_c3_a3_d3_e$.

Lastly, when $C = 6$ there is a single vertex color configuration:
$3_a3_b3_c3_d3_e3_f$.

3.2 Finding the Edge-c-Colorings Admitted by a Vertex Color Configuration

To find all uniform edge-c-colorings admitted by a given vertex color configuration, Lemma 1 guarantees that it is sufficient to find all edge-colorings of a

Table 1: vertex color configurations of type 3^6

$3_a3_a3_a3_a3_a3_a$	$3_a3_a3_b3_a3_a3_b$	$3_a3_a3_b3_c3_b3_c$	$3_a3_b3_a3_c3_a3_d$
$3_a3_a3_a3_a3_a3_b$	$3_a3_a3_b3_a3_a3_c$	$3_a3_a3_b3_c3_b3_d$	$3_a3_b3_a3_c3_b3_c$
$3_a3_a3_a3_a3_b3_b$	$3_a3_a3_b3_a3_b3_b$	$3_a3_a3_b3_c3_c3_b$	$3_a3_b3_a3_c3_b3_d$
$3_a3_a3_a3_a3_b3_c$	$3_a3_a3_b3_a3_b3_c$	$3_a3_a3_b3_c3_c3_d$	$3_a3_b3_a3_c3_d3_c$
$3_a3_a3_a3_b3_a3_b$	$3_a3_a3_b3_a3_c3_b$	$3_a3_a3_b3_c3_d3_b$	$3_a3_b3_a3_c3_d3_e$
$3_a3_a3_a3_b3_a3_c$	$3_a3_a3_b3_a3_c3_c$	$3_a3_a3_b3_c3_d3_e$	$3_a3_b3_c3_a3_b3_c$
$3_a3_a3_a3_b3_b3_b$	$3_a3_a3_b3_a3_c3_d$	$3_a3_b3_a3_b3_a3_b$	$3_a3_b3_c3_a3_b3_d$
$3_a3_a3_a3_b3_b3_c$	$3_a3_a3_b3_b3_c3_c$	$3_a3_b3_a3_b3_a3_c$	$3_a3_b3_c3_a3_d3_e$
$3_a3_a3_a3_b3_c3_b$	$3_a3_a3_b3_b3_c3_d$	$3_a3_b3_a3_b3_c3_d$	$3_a3_b3_c3_d3_e3_f$
$3_a3_a3_a3_b3_c3_d$			

Table 2: vertex color configurations of type 3.4.6.4

$3_a4_a6_a4_a$	$3_a4_a6_b4_b$	$3_a4_b6_a4_c$	$3_a4_b6_c4_a$
$3_a4_a6_a4_b$	$3_a4_a6_b4_c$	$3_a4_b6_b4_a$	$3_a4_b6_c4_d$
$3_a4_a6_b4_a$	$3_a4_b6_a4_b$	$3_a4_b6_b4_c$	

Table 3: vertex color configurations of type 3.12^2

$3_a12_a12_a$	$3_a12_a12_b$	$3_a12_b12_a$	$3_a12_b12_c$

Table 4: vertex color configurations of type 4^4

$4_a4_a4_a4_a$	$4_a4_a4_b4_b$	$4_a4_b4_a4_b$	$4_a4_b4_c4_d$
$4_a4_a4_a4_b$	$4_a4_a4_b4_c$	$4_a4_b4_a4_c$	

period parallelogram of the underlying Archimedean tiling. The uniform colorings so generated can then be detected using Lemma 2. However, in practice the process of enumerating all uniform edge-c-colorings admitted by a given vertex color configuration can almost always be accomplished more efficiently. Typically, the colorings are found by starting with a "blank" tiling, coloring the edges surrounding a few adjacent vertices, and seeing what is forced by the geometry and symmetry of the underlying tiling. It is almost always the case that very few options are possible.

To illustrate this, consider the vertex color configuration $4_a4_b4_a4_c$. The process given here is representative of how the remaining vertex color configurations are handled. At left in Fig. 5, a random vertex of the Archimedean tiling (4^4) labeled P has been colored with configuration $4_a4_b4_a4_c$. With this

Table 5: vertex color configurations of type 4.8^2

$4_a8_a8_a$	$4_a8_a8_b$	$4_a8_b8_a$	$4_a8_b8_c$

Table 6: vertex color configurations of type $3^4.6$

$3_a3_a3_a3_a6_a$	$3_a3_a3_b3_a6_c$	$3_a3_a3_b3_c6_c$	$3_a3_b3_a3_b6_c$	$3_a3_b3_c3_b6_b$
$3_a3_a3_a3_a6_b$	$3_a3_a3_b3_b6_a$	$3_a3_a3_b3_c6_d$	$3_a3_b3_a3_c6_c$	$3_a3_b3_c3_a6_d$
$3_a3_a3_a3_b6_b$	$3_a3_a3_b3_b6_b$	$3_a3_b3_a3_a6_a$	$3_a3_b3_a3_c6_d$	$3_a3_b3_c3_b6_d$
$3_a3_a3_a3_b6_c$	$3_a3_a3_b3_b6_c$	$3_a3_b3_a3_a6_b$	$3_a3_b3_c3_a6_a$	$3_a3_b3_c3_c6_d$
$3_a3_a3_b3_a6_a$	$3_a3_a3_b3_c6_a$	$3_a3_b3_a3_a6_c$	$3_a3_b3_c3_a6_b$	$3_a3_b3_c3_d6_d$
$3_a3_a3_b3_a6_b$	$3_a3_a3_b3_c6_b$	$3_a3_b3_a3_b6_b$	$3_a3_b3_c3_a6_c$	$3_a3_b3_c3_d6_e$

Table 7: vertex color configurations of type $3^3.4^2$

$3_a3_a3_a4_a4_a$	$3_a3_a3_b4_b4_b$	$3_a3_b3_a4_b4_a$	$3_a3_b3_b4_a4_c$	$3_a3_b3_c4_a4_d$
$3_a3_a3_a4_a4_b$	$3_a3_a3_b4_b4_c$	$3_a3_b3_a4_b4_c$	$3_a3_b3_b4_b4_c$	$3_a3_b3_c4_b4_d$
$3_a3_a3_a4_b4_a$	$3_a3_a3_b4_c4_b$	$3_a3_b3_a4_c4_a$	$3_a3_b3_b4_c4_a$	$3_a3_b3_c4_c4_c$
$3_a3_a3_a4_b4_b$	$3_a3_a3_b4_c4_d$	$3_a3_b3_a4_c4_b$	$3_a3_b3_b4_c4_c$	$3_a3_b3_c4_c4_d$
$3_a3_a3_a4_b4_c$	$3_a3_b3_a4_a4_a$	$3_a3_b3_a4_c4_c$	$3_a3_b3_b4_c4_d$	$3_a3_b3_c4_d4_c$
$3_a3_a3_b4_a4_b$	$3_a3_b3_a4_a4_b$	$3_a3_b3_a4_c4_d$	$3_a3_b3_c4_a4_c$	$3_a3_b3_c4_d4_e$
$3_a3_a3_b4_a4_c$	$3_a3_b3_a4_a4_c$	$3_a3_b3_b4_a4_a$		

Table 8: vertex color configurations of type $3^2.4.3.4$

$3_a3_a4_a3_a4_a$	$3_a3_a4_b3_b4_a$	$3_a3_b4_a3_a4_b$	$3_a3_b4_a3_c4_d$	$3_a3_b4_c3_b4_c$
$3_a3_a4_a3_a4_b$	$3_a3_a4_b3_b4_b$	$3_a3_b4_a3_a4_c$	$3_a3_b4_b3_b4_b$	$3_a3_b4_c3_b4_d$
$3_a3_a4_a3_b4_a$	$3_a3_a4_b3_b4_c$	$3_a3_b4_a3_b4_b$	$3_a3_b4_b3_b4_c$	$3_a3_b4_c3_c4_b$
$3_a3_a4_a3_b4_b$	$3_a3_a4_b3_c4_b$	$3_a3_b4_a3_b4_c$	$3_a3_b4_b3_c4_b$	$3_a3_b4_c3_c4_d$
$3_a3_a4_a3_b4_c$	$3_a3_a4_b3_c4_c$	$3_a3_b4_a3_c4_b$	$3_a3_b4_b3_c4_c$	$3_a3_b4_c3_d4_b$
$3_a3_a4_b3_a4_b$	$3_a3_a4_b3_c4_d$	$3_a3_b4_a3_c4_c$	$3_a3_b4_b3_c4_d$	$3_a3_b4_c3_d4_e$
$3_a3_a4_b3_a4_c$				

choice of aspect for P, any other vertex in an edge-3-coloring containing P may be in one of two aspects (blue up or blue down). Notice that the aspects of the vertices on the vertical line containing P are completely determined (right, Fig. 5). Thus, any edge-3-coloring of type $(4_a4_b4_a4_c)$ corresponds to a choice of aspects of the vertices on the horizontal line containing P (Fig. 6). *Any* such choice yields an edge-3-coloring, and thus there are uncountably many edge-3-colorings of type $(4_a4_b4_a4_c)$. Some of these coloings are periodic,

Table 9: vertex color configurations of type 3.6.3.6

$3_a6_a3_a6_a$ $3_a6_a3_a6_b$ $3_a6_a3_b6_b$	$3_a6_a3_b6_c$ $3_a6_b3_a6_b$	$3_a6_b3_a6_c$ $3_a6_b3_b6_a$	$3_a6_b3_c6_a$ $3_a6_b3_c6_d$

Table 10: vertex color configurations of type 4.6.12

$4_a6_a12_a$	$4_a6_a12_b$	$4_a6_b12_a$	$4_a6_b12_b$	$4_a6_b12_c$

Table 11: vertex color configurations of type 6^3

$6_a6_a6_a$	$6_a6_a6_b$	$6_a6_b6_c$

but by Lemma 1, only those with small periods may be uniform. In this particular case the small period can be quickly determined: On the horizontal line containing P, there cannot be more than two consecutive vertices in the same aspect, for if there were vertices A, B, C, and D on this line where A, B, and C are in the same aspect and D is in a different aspect, then there is not symmetry of the coloring taking D to B. Thus, the possible patterns of aspects on the line containing P are limited to

$$\dots \alpha, \alpha, \alpha, \dots,$$

$$\dots, \alpha, \beta, \alpha, \beta, \dots,$$

or

$$\dots, \alpha, \alpha, \beta, \beta, \alpha, \alpha, \beta, \beta, \dots.$$

It is easily checked (using Lemma 2) that these three patterns all correspond to uniform edge-3-colorings of type $(4_a4_b4_a4_c)$, as depicted in Fig. 7.

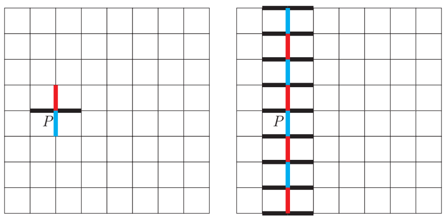


Fig. 5: The coloring of P determines the coloring surrounding vertices above and below P.

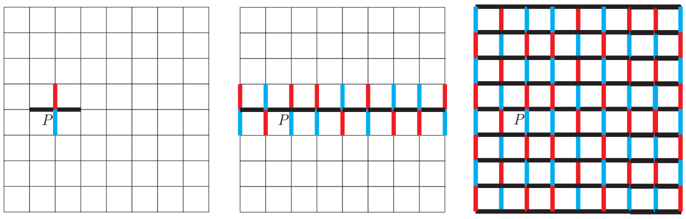


Fig. 6: A choice of aspects on the horizontal containing P determines the coloring of the tiling. This edge-3-coloring is not uniform.

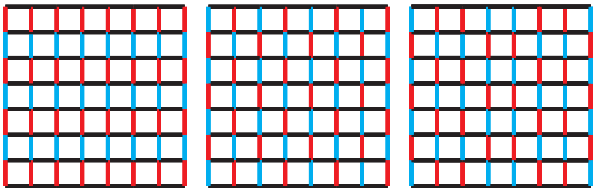


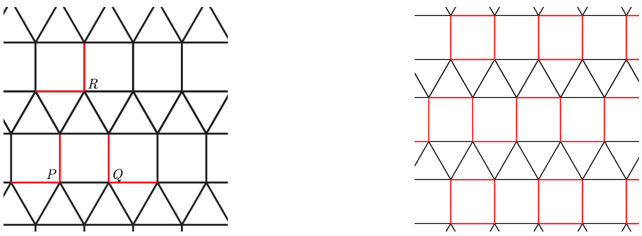
Fig. 7: The three edge-3-colorings of type $(4_a4_b4_a4_c)$

4 Nonuniform Archimedean Coloring

As was demonstrated earlier (Fig. 4), some vertex color configurations admit nonuniform Archimedean colorings. While every occurrence of nonunformity was not cataloged, the occurrences observed were all very similar. To illustrate the mechanism, consider the vertex color configuration $4_a4_b4_a4_c$, as in Fig. 6. After placing an initial copy of $4_a4_b4_a4_c$ so that the edges marked a are oriented horizontally, the other vertices along this horizontal line must be assigned a vertex color configuration with the edges marked a oriented horizontally as well; thus, the other two colors (b and c) must be oriented vertically, and either choice of orientation (b up or b down) is possible. Once a choice has been made for each vertex along our initial horizontal line, the rest of the coloring is determined. So, the set of all Archimedean colorings of type $(4_a4_b4_a4_c)$ correspond the set of vertices along a horizontal line, which can be thought of as a binary digits, and the whole line can be thought of as a bi-infinite string of 0's and 1's. Clearly there are uncountably many such strings.

The vertex color configuration $4_a4_b4_a4_c$ admits both uniform and nonuniform Archimedean edge colorings, but a few vertex color configurations admit nonuniform Archimedean edge colorings but no uniform edge colorings. For example, consider $3_a3_a3_a4_b4_b$. In Fig. 8a, adjacent vertices P and Q lying at the ends of an edge between a triangle and a square have been chosen and their incident edges colored as $3_a3_a3_a4_b4_b$. Without loss of generality, suppose that $\overline{PQ}$ has been assigned color a (black). The only color-preserving symmetry of that takes P to Q is a reflection γ across the perpendicular bisector of $\overline{PQ}$. But, by considering the effect of γ on the horizontal edges incident to point R, exactly one of which must be colored b (red), we see that γ is not a color-preserving symmetry of any Archimedean coloring in which every vertex has color configuration $3_a3_a3_a4_b4_b$. There are, however, uncountably many nonuniform Archimedean colorings of this type as each row of squares can be be colored in independent ways. In Fig. 8b is such a coloring.

Several other vertex color configurations admit uncountably many nonuniform Archimedean edge colorings as well. We report those found in Table 12.



(a) The only possible color-preserving symmetry from P to Q must fix R.

(b) A nonuniform Archimedean edge-2-coloring of type $(3_a3_a3_a4_b4_b)$.

Fig. 8

Table 12: Some vertex color configurations admitting uncountably many nonuniform Archimedean edge colorings. x and y represent wildcards. Those marked with $*$ admit no uniform edge colorings.

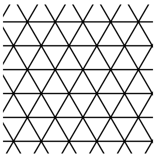
$3_a3_a3_a3_a3_a3_b$	$3_a3_b3_x4_y4_x$	$3_a3_b3_c4_c4_c$	$3_a3_a3_b4_a4_c^*$
$3_a3_a3_a4_b4_c^*$	$3_a3_a3_b4_b4_c^*$	$3_a3_a3_a4_b4_b^*$	$4_a4_a4_a4_b$
$4_a4_b4_a4_c$	$4_a4_b4_c4_d$	$4_a8_a8_b$	$4_a8_b8_b^*$
$4_a4_b8_c$	$6_a6_a6_b$	$6_a6_b6_c$	

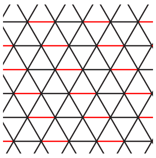
5 The 109 Uniform Edge-c-Colorings

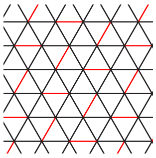
5.1 Figures of the 109 Uniform Edge Colorings

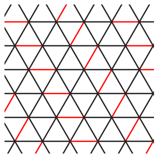
Table 13: vertex color configurations admitting uniform edge-c-colorings

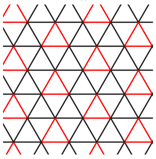
Vertex Color Configuration	Number of Colorings	Vertex Color Configuration	Number of Colorings
$3_a3_a3_a3_a3_a3_a$	1	$3_a3_a3_a3_a6_a$	1
$3_a3_a3_a3_a3_a3_b$	3	$3_a3_a3_a3_b6_b$	1
$3_a3_a3_a3_a3_b3_b$	2	$3_a3_a3_b3_a6_a$	1
$3_a3_a3_a3_a3_b3_c$	2	$3_a3_a3_b3_b6_b$	1
$3_a3_a3_a3_b3_a3_b$	1	$3_a3_a3_b3_c6_c$	1
$3_a3_a3_a3_b3_a3_c$	2	$3_a3_a3_a4_a4_a$	1
$3_a3_a3_b3_a3_a3_b$	1	$3_a3_a3_a4_b4_a$	1
$3_a3_a3_b3_a3_a3_c$	1	$3_a3_a3_b4_a4_b$	1
$3_a3_a3_b3_a3_b3_b$	3	$3_a3_a3_b4_b4_b$	1
$3_a3_a3_b3_a3_b3_c$	2	$3_a3_a3_b4_c4_b$	1
$3_a3_a3_b3_a3_c3_b$	4	$3_a3_b3_a4_a4_a$	2
$3_a3_a3_b3_a3_c3_c$	2	$3_a3_b3_a4_b4_a$	2
$3_a3_a3_b3_a3_c3_d$	1	$3_a3_b3_a4_c4_a$	2
$3_a3_a3_b3_b3_c3_c$	1	$3_a3_b3_c4_a4_c$	2
$3_a3_a3_b3_c3_c3_b$	1	$3_a3_b3_c4_c4_c$	2
$3_a3_a3_b3_c3_d3_b$	2	$3_a3_b3_c4_d4_c$	2
$3_a3_b3_a3_c3_b3_c$	1	$3_a3_a4_a3_a4_a$	1
$3_a3_b3_a3_c3_b3_d$	2	$3_a3_a4_a3_b4_b$	1
$3_a3_b3_c3_a3_b3_c$	1	$3_a3_a4_b3_a4_b$	1
$3_a3_b3_c3_a3_b3_d$	1	$3_a3_b4_b3_b4_b$	1
$3_a3_b3_c3_a3_d3_e$	1	$3_a3_b4_b3_c4_c$	1
$3_a4_a6_a4_a$	1	$3_a3_b4_c3_b4_c$	1
$3_a4_a6_b4_a$	1	$3_a6_a3_a6_a$	1
$3_a4_b6_b4_a$	1	$3_a6_b3_b6_a$	1
$3_a4_b6_c4_a$	1	$4_a6_a12_a$	1
$3_a12_a12_a$	1	$4_a6_a12_b$	1
$3_a12_b12_a$	1	$4_a6_b12_a$	1
$4_a4_a4_a4_a$	1	$4_a6_b12_b$	1
$4_a4_a4_a4_b$	3	$4_a6_b12_c$	1
$4_a4_a4_b4_b$	4	$6_a6_a6_a$	1
$4_a4_a4_b4_c$	4	$6_a6_a6_b$	3
$4_a4_b4_a4_b$	1	$6_a6_b6_c$	3
$4_a4_b4_a4_c$	3	Total	109
$4_a4_b4_c4_d$	4		
$4_a8_a8_a$	1		
$4_a8_a8_b$	2		
$4_a8_b8_a$	1		
$4_a8_b8_c$	2		

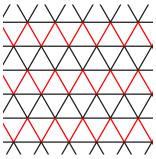

(3_a^6)

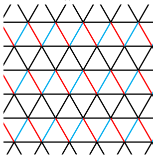

$(3_a^5 3_b)_1$

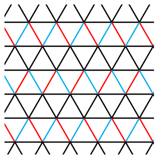

$(3_a^5 3_b)_2$

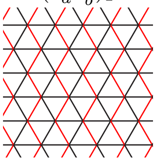

$(3_a^5 3_b)_3$

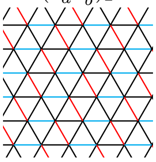

$(3_a^4 3_b^2)_1$

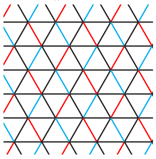

$(3_a^4 3_b^2)_2$

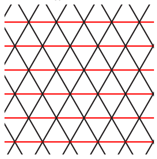

$(3_a^3 3_b 3_c)_1$

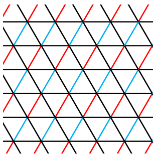

$(3_a^3 3_b 3_c)_2$

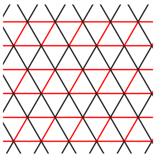

$(3_a^3 3_b 3_a 3_b)$

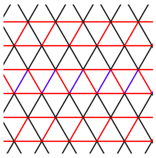

$(3_a^3 3_b 3_a 3_c)_1$

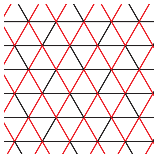

$(3_a^3 3_b 3_a 3_c)_2$

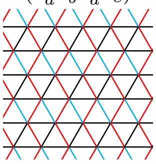

$(3_a^2 3_b 3_a^2 3_b)$

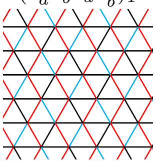

$(3_a^2 3_b 3_a^2 3_c)$

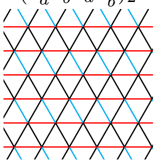

$(3_a^2 3_b 3_a 3_b^2)_1$

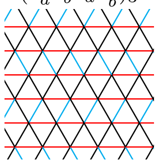

$(3_a^2 3_b 3_a 3_b^2)_2$

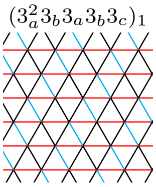

$(3_a^2 3_b 3_a 3_b^2)_3$

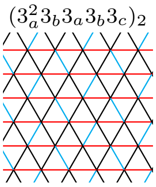

$(3_a^2 3_b 3_a 3_b 3_c)_1$

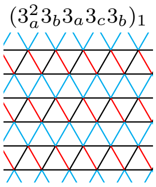

$(3_a^2 3_b 3_a 3_b 3_c)_2$

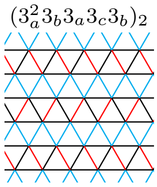

$(3_a^2 3_b 3_a 3_c 3_b)_1$

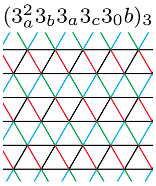

$(3_a^2 3_b 3_a 3_c 3_b)_2$

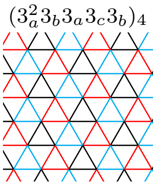

$(3_a^2 3_b 3_a 3_c 3_0 b)_3$

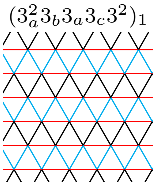

$(3_a^2 3_b 3_a 3_c 3_b)_4$

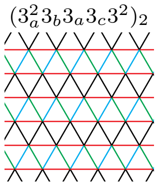

$(3_a^2 3_b 3_a 3_c 3^2)_1$

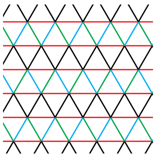

$(3_a^2 3_b 3_a 3_c 3^2)_2$

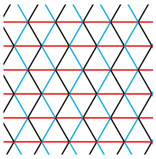

$(3_a^2 3_b 3_a 3_c 3_d)$

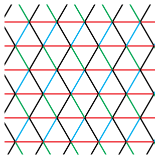

$(3_a^2 3_b^2 3_c^2)$

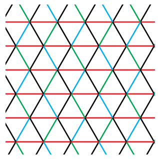

$(3_a^2 3_b 3_c^2 3_b)$

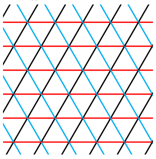

$(3_a^2 3_b 3_c 3_d 3_b)_1$

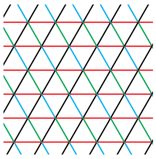

$(3_a^2 3_b 3_c 3_d 3_b)_2$

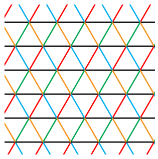

$(3_a 3_b 3_a 3_c 3_b 3_c)$

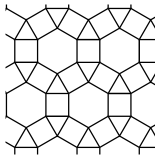

$(3_a 3_b 3_a 3_c 3_b 3_d)_1$

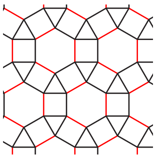

$(3_a 3_b 3_a 3_c 3_b 3_d)_2$

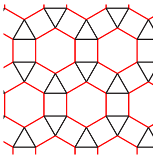

$(3_a 3_b 3_c 3_a 3_b 3_c)$

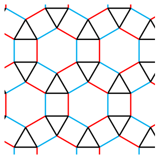

$(3_a 3_b 3_c 3_a 3_b 3_d)$

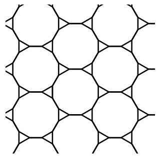

$(3_a 3_b 3_c 3_a 3_d 3_e)$

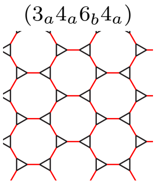

$(3_a 4_a 6_a 4_a)$

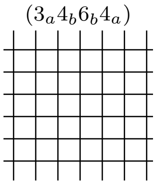

$(3_a 4_a 6_b 4_a)$

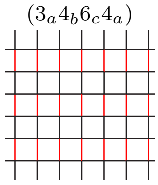

$(3_a 4_b 6_b 4_a)$

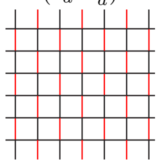

$(3_a 4_b 6_c 4_a)$

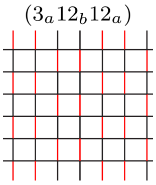

$(3_a 12_a^2)$

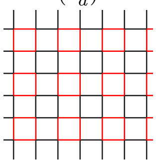

$(3_a 12_b 12_a)$

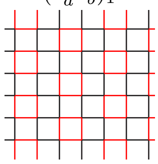

(4_a^4)

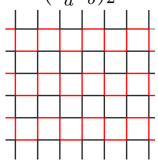

$(4_a^3 4_b)_1$

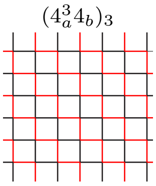

$(4_a^3 4_b)_2$

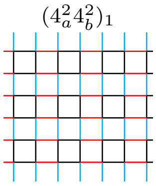

$(4_a^3 4_b)_3$

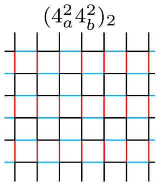

$(4_a^2 4_b^2)_1$

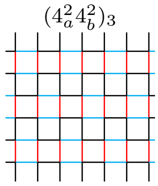

$(4_a^2 4_b^2)_2$

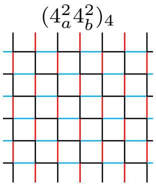

$(4_a^2 4_b^2)_3$

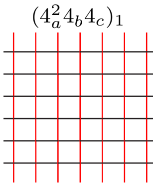

$(4_a^2 4_b^2)_4$

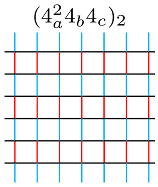

$(4_a^2 4_b 4_c)_1$

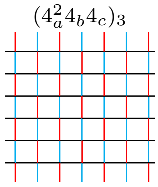

$(4_a^2 4_b 4_c)_2$

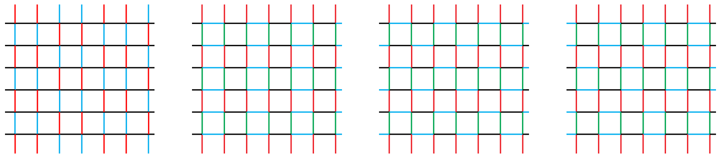

$(4_a^2 4_b 4_c)_3$


$(4_a^2 4_b 4_c)_4$

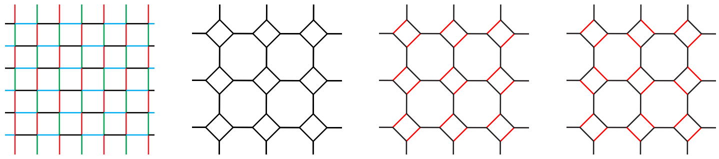

$(4_a 4_b 4_a 4_b)$


$(4_a 4_b 4_a 4_c)_1$

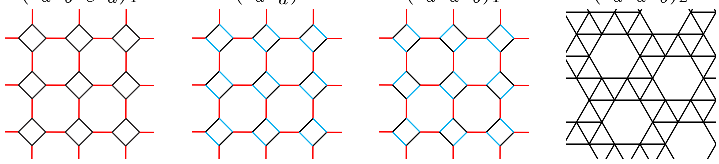

$(4_a 4_b 4_a 4_c)_2$



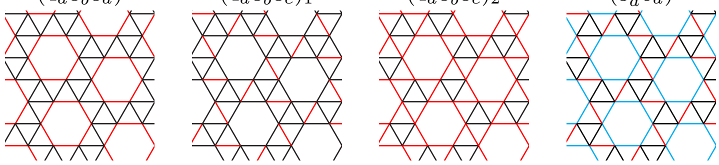
$(4_a4_b4_a4_c)_3$ $(4_a4_b4_c4_d)_1$ $(4_a4_b4_c4_d)_2$ $(4_a4_b4_c4_d)_3$



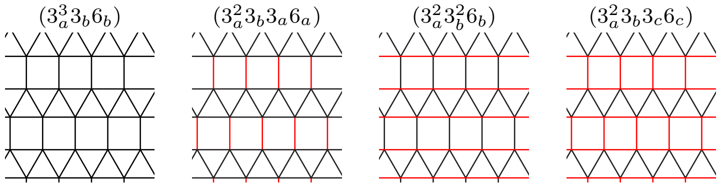
$(4_a4_b4_c4_d)_4$ $(4_a8_a^2)$ $(4_a8_a8_b)_1$ $(4_a8_a8_b)_2$



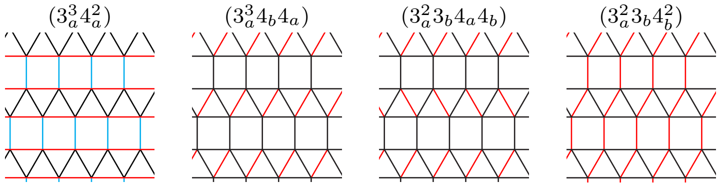
$(4_a8_b8_a)$ $(4_a8_b8_c)_1$ $(4_a8_b8_c)_2$ $(3_a^4 6_a)$



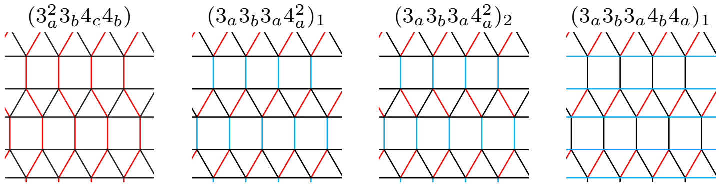
$(3_a^3 3_b 6_b)$ $(3_a^2 3_b 3_a 6_a)$ $(3_a^2 3_b^2 6_b)$ $(3_a^2 3_b 3_c 6_c)$



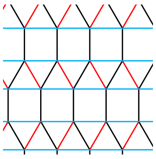
$(3_a^3 4_a^2)$ $(3_a^3 4_b 4_a)$ $(3_a^2 3_b 4_a 4_b)$ $(3_a^2 3_b 4_b^2)$

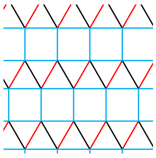


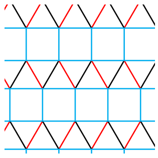
$(3_a^2 3_b 4_c 4_b)$ $(3_a 3_b 3_a 4_a^2)_1$ $(3_a 3_b 3_a 4_a^2)_2$ $(3_a 3_b 3_a 4_b 4_a)_1$

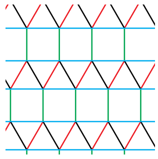


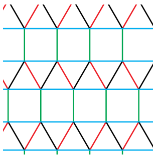
$(3_a 3_b 3_a 4_b 4_a)_2$ $(3_a 3_b 3_a 4_c 4_a)_1$ $(3_a 3_b 3_a 4_c 4_a)_2$ $(3_a 3_b 3_c 4_a 4_c)_1$

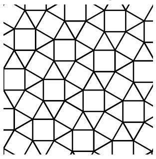

$(3_a3_b3_c4_a4_c)_2$

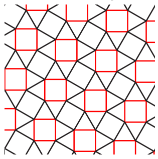

$(3_a3_b3_c4_c^2)_1$

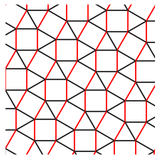

$(3_a3_b3_c4_c^2)_2$

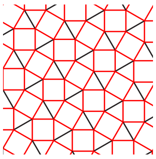

$(3_a3_b3_c4_d4_c)_1$

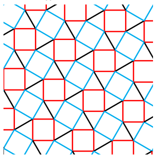

$(3_a3_b3_c4_d4_c)_2$

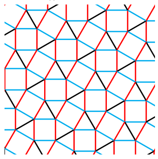

$(3_a^24_a3_a4_a)$

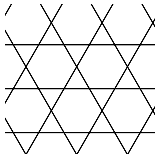

$(3_a^24_a3_b4_b)$

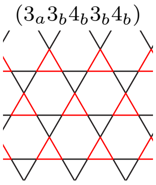

$(3_a^24_b3_a4_b)$

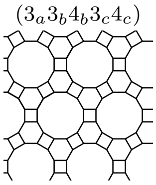

$(3_a3_b4_b3_b4_b)$

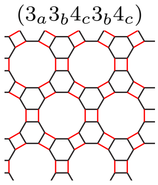

$(3_a3_b4_b3_c4_c)$

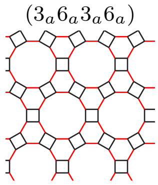

$(3_a3_b4_c3_b4_c)$

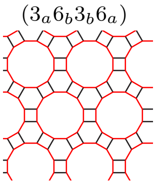

$(3_a6_a3_a6_a)$

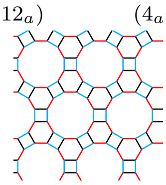

$(3_a6_b3_b6_a)$

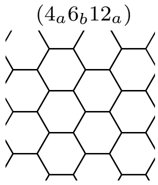

$(4_a6_a12_a)$

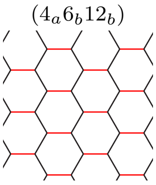

$(4_a6_a12_b)$

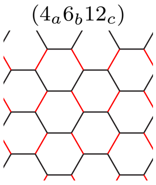

$(4_a6_b12_a)$

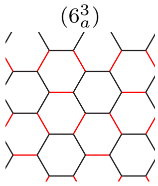

$(4_a6_b12_b)$

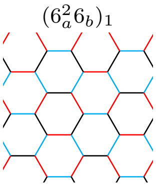

$(4_a6_b12_c)$

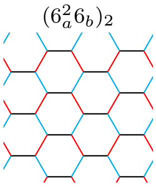

(6_a^3)

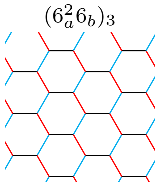

$(6_a^26_b)_1$


$(6_a^26_b)_2$


$(6_a^26_b)_3$


$(6_a6_b6_c)_1$


$(6_a6_b6_c)_2$


$(6_a6_b6_c)_3$

References

1. Grünbaum, B., Shephard, G.C.: Tilings and Patterns, Freeman, New York (1987)