Journal of Knot Theory and Its Ramifications
Vol. 21, No. 14 (2012) 1250120 (15 pages)

DOI: 10.1142/S0218216512501209

THE STICK NUMBER FOR THE SIMPLE HEXAGONAL LATTICE

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Received 19 May 2011
Accepted 28 July 2012
Published 9 October 2012

ABSTRACT

This work is motivated by a paper of Huh and Oh, in which the authors prove that the minimum number of sticks required to form a knot in $\mathbb{Z}^3$ is 12. In this article the authors prove that the stick number in the simple hexagonal lattice is 11. Moreover, the stick number of the trefoil in the simple hexagonal lattice is 11.

Keywords: Knots; polygons; lattice; stick number; stick index; simple hexagonal lattice.

Mathematics Subject Classification 2010: 57M25

1. Stick Knots in the Simple Hexagonal Lattice

The *simple hexagonal lattice* (*sh*) is the point lattice with basis $\{x, y, w\}$ where $x = \langle 1, 0, 0\rangle$, $y = \langle 1/2, \sqrt{3}/2, 0\rangle$ and $w = \langle 0, 0, 1\rangle$. The *directions* of sh are the vectors x, y, w and $z = y - x$. An *sh-stick* is a straight line segment with endpoints in sh that is parallel to a direction of sh. An sh-stick that is parallel to x will be called an *x-stick* and y-, z-, and w-sticks are defined in the same way. In Fig. 1 we have an 11-stick 3_1. Letting $X = -x,\ Y = -y,\ Z = -z$ and $W = -w$, the depicted knot can be described by the string of vectors $wxxxWWzzwwwYYYWWxyyyyXXYYY$. The minimum number of sh-sticks required to form a nontrivial knot is the *stick number* of sh. The minimum number of sh-sticks required to form a knot of type K is the *sh-stick number of K*, denoted $S_{\rm sh}(K)$.

Given a polygon $\mathcal{P}$ made from sh-sticks, $|\mathcal{P}|$ will denote the number of sh-sticks in $\mathcal{P}$. $|\mathcal{P}|_x$ denotes the number of x-sticks in $\mathcal{P}$ with $|\mathcal{P}|_y$, $|\mathcal{P}|_z$ and $|\mathcal{P}|_w$ defined similarly. Planes that are parallel to the xy-plane and contain x-, y- or z- sticks of $\mathcal{P}$ will be called *w-levels of $\mathcal{P}$*. A polygon $\mathcal{P}$ is *properly leveled with respect to w* if each w-level of $\mathcal{P}$ contains just one connected polygonal arc. Note that if $\mathcal{P}$ is properly

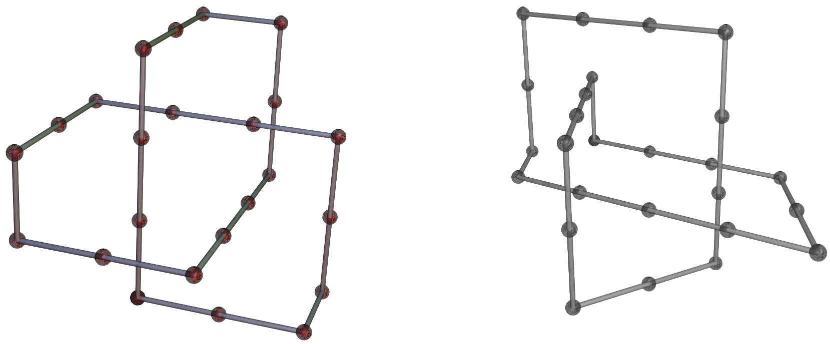

Fig. 1. At left is a 12-stick 3_1 knot in $\mathbb{Z}^3$. At right is an 11-stick 3_1 knot in the simple hexagonal lattice (sh).

leveled with respect to w and has n w-levels, then $|\mathcal{P}|_w = n$. A *trivial polygon* is a trivial polygonal knot. Two polygons are said to be *equivalent* if they are ambient isotopic in $\mathbb{R}^3$. A polygon $\mathcal{P}$ is called *reducible* if there is another equivalent polygon that has fewer sticks; otherwise *irreducible*. If $\mathcal{P}$ has n w-levels, then we enumerate the w-levels, calling them levels $1, 2, \ldots, n$ (like heights). In particular, w-levels 1 and n are the *boundary w-levels*.

We give a few lemmas (Lemmas 1.1, 1.2, 1.3, and 1.4) that are analogous to lemmas given in [1] (that apply to the cubic lattice) and whose proofs are identical. These lemmas are stated only for w-sticks as w-sticks are perpendicular to the xy-plane and so behave like sticks in the cubic lattice.

Lemma 1.1. *For a given polygon $\mathcal{P}$, there is a polygon $\mathcal{P}'$ of the same knot type that is properly leveled with respect to w such that $|\mathcal{P}'| = |\mathcal{P}|$.*

The basic idea of the proof of this lemma is that if some w-level, say the mth, contains more than one connected arc, one can move the part of $\mathcal{P}$ above (or below) the mth w-level up (or down) by one or more levels (by stretching the w sticks). This clears out room to move one or more of the arcs in the mth level onto other levels so that each connected arc is on its own w-level.

For the remainder of the paper, we will assume that all polygons are properly leveled with respect to w.

Lemma 1.2. *If two w-sticks of a polygon $\mathcal{P}$ have their endpoints on the same two w-levels, then $\mathcal{P}$ must be the trivial polygon.*

Lemma 1.3. *Let $\mathcal{P}$ be a nontrivial irreducible polygon. Then the boundary levels of $\mathcal{P}$ contain only one or two connected sticks. Furthermore, any w-stick of $\mathcal{P}$ with an endpoint on a boundary level has length at least two.*

Lemma 1.4. *If $\mathcal{P}$ is a nontrivial irreducible polygon, then $|\mathcal{P}|_w \geq 4$.*

Lemma 1.5. *If $|\mathcal{P}| \leq 10$ and $|\mathcal{P}|_w \geq 5$, then $\mathcal{P}$ is a trivial polygon.*

Proof. First note that since $\mathcal{P}$ is properly leveled with respect to w, it must have at least one sh-stick on each w-level. Also, $\mathcal{P}$ must have the same number of w-sticks as w-levels. If $\mathcal{P}$ were to have six or more w-levels, then $\mathcal{P}$ must have at least twelve sh-sticks; thus we can exclude the possibility that $|\mathcal{P}|_w \geq 6$.

If $\mathcal{P}$ has five w-levels, then it must have exactly five w-sticks and exactly one sh-stick on each w-level. We will show that $\mathcal{P}$ has a projection having two or fewer crossings, implying that $\mathcal{P}$ is a trivial polygon. Let $\pi(\mathcal{P})$ be the perpendicular projection in the w direction of $\mathcal{P}$ onto any w-level. By Lemma 1.3, the w-sticks of $\mathcal{P}$ project to five distinct points in $\pi(\mathcal{P})$. But, because $\mathcal{P}$ is made from straight lines, it is possible that entire line segments from $\mathcal{P}$ that lie in different w-levels will be projected on top of each other, so we may have to slightly isotope the knot before projecting to reveal *obscured crossings*. If $\pi(\mathcal{P})$ contains an *unobscured crossing* (i.e. a crossing forming an angle of $\pi/3$ in $\pi(\mathcal{P})$), then by rotating or reflecting $\mathcal{P}$ appropriately and giving an orientation to $\mathcal{P}$, we can say without loss of generality that a crossing is seen in $\pi(\mathcal{P})$ in one of two possible ways, which are depicted at left in Fig. 2. In either case, it must be checked that any possible way to complete the circuit in $\pi(\mathcal{P})$ (so that this projection will represent the closed polygon $\mathcal{P}$) gives a projection that has at most two crossings.

In the first case, (upper left of Fig. 2), because of the orientation on $\mathcal{P}$ we must connect point B to point C and we must connect point D to point A with the remaining three lattice sticks. If B can be connected to C with just one sh-stick, no new crossings will be formed in connecting them and there will remain two sh-sticks with which to connect D to A. D can be connected to A using two sh-sticks in a few different ways. We depict representative ways in which new crossings may occur in the top middle and top right diagrams of Fig. 2. As illustrated in the top right diagram in Fig. 2, in connecting D to A with two sticks, the projections of one or both of those sticks may be obscured by the segments AB or CD in $\pi(\mathcal{P})$. Lastly, it is seen that for any configuration, by first slightly perturbing the w-sticks of $\mathcal{P}$ at A, B, C, D and E without introducing crossings (as in Fig. 3), a projection of a slightly isotoped version of $\mathcal{P}$ with at most two crossings is possible.

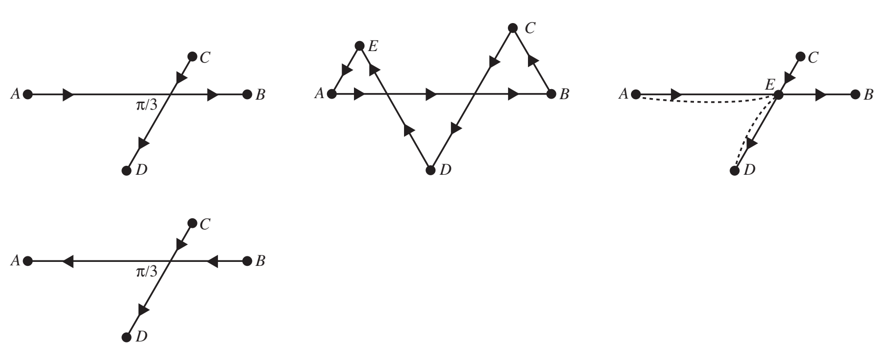

Fig. 2. A crossing in $\pi(\mathcal{P})$.

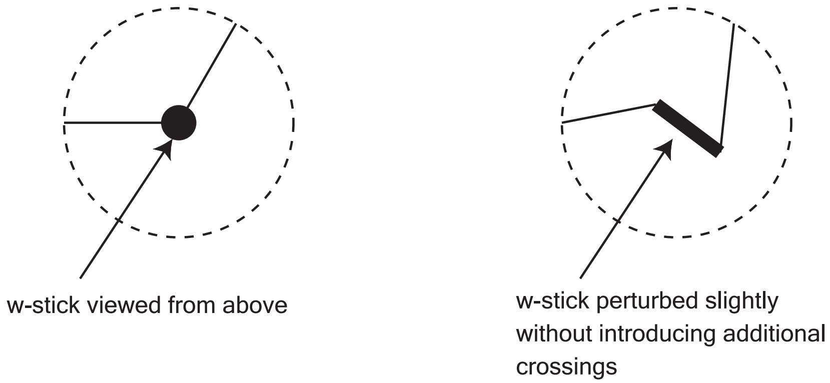

Fig. 3. Perturbing the w-sticks of $\mathcal{P}$ without introducing crossings.

In the second case (lower left, Fig. 2), it is impossible to complete the circuit with only three sticks.

If in $\pi(\mathcal{P})$ there are no unobscured crossings and no line segments have been projected on top of each other, then using isotopies as depicted in Fig. 3, we see that $\mathcal{P}$ is trivial. But, even if some line segments have been projected on top of each other, potentially contributing obscured crossings, we will show that $\mathcal{P}$ can be slightly isotoped to obtain a projection $\pi(\mathcal{P})$ with at most one crossing. To prove this, assume that some line segments have projected on top of each other in $\pi(\mathcal{P})$ and that there are no unobscured crossings in $\pi(\mathcal{P})$. Let W be the maximum number of w-sticks whose projection lie on a single line in $\pi(\mathcal{P})$.

If $W = 5$, then $\mathcal{P}$ lies in a single plane and so must be a trivial knot.

In the case that $W = 4$, let A be the projection of the w-stick that is not collinear with the other four. Viewing $\pi(\mathcal{P})$ as a graph, each of the five w-stick projections must have valence 2. Thus, A connects to two other points in $\pi(\mathcal{P})$; we mark these points with an "X" in Fig. 4. Up to symmetry, there are only four distinct possibilities for the locations of the points marked X. For each of these, there are two ways to connect the five points to form a closed loop. The top row of Fig. 4

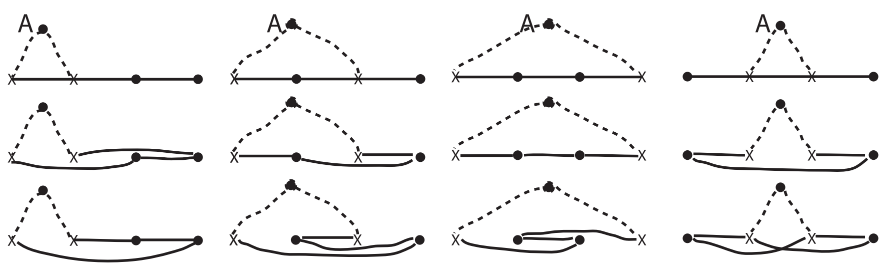

Fig. 4. Four collinear w-stick projections.

shows the four possible arrangements of the points marked X, and below each of these four arrangements are slightly isotoped versions of the possible ways to form a closed loop through the five points. Each picture has one crossing or zero crossings.

Suppose that $W = 3$. If there is only one set of three collinear w-stick projections, then since we are under the hypothesis that there are no unobscured crossings, the other two w-stick projections must lie on one side of the line through the three collinear w-stick projections. There are two ways (up to symmetry) that this situation can occur, and via a slight isotopy it is easily seen that both of these possibilities contain no obscured crossings (Fig. 5). Notice that since we are under the hypothesis that there are no unobscured crossings, we know that no crossings occur outside of a small neighborhood of the line segment containing the three collinear points (shaded grey in Fig. 5).

If there are two sets of three collinear w-stick projections, we see in Fig. 6 the isotopies that reveal that there is a projection of $\mathcal{P}$ with no crossings.

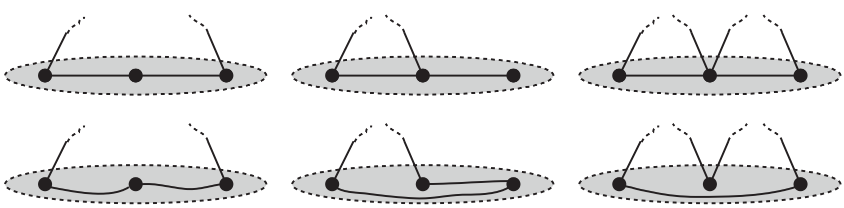

Fig. 5. One set of three collinear w-stick projections.

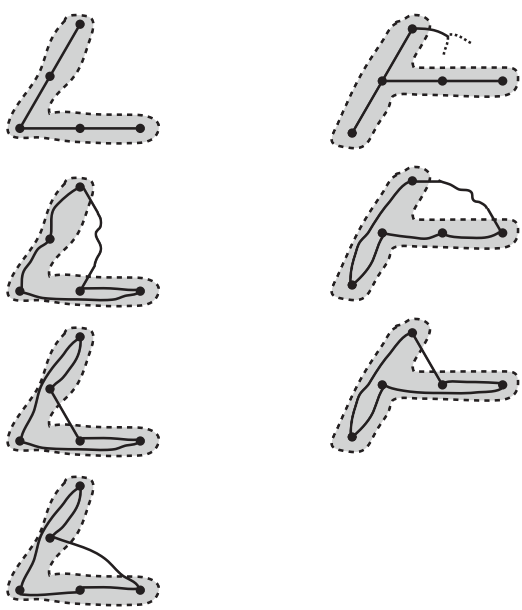

Fig. 6. Two sets of three collinear w-stick projections.

For $W = 2$, $\pi(\mathcal{P})$ will be a simple polygon, and by perturbing the w-sticks of $\mathcal{P}$ as in Fig. 3, we see that $\mathcal{P}$ may be projected to a diagram with no crossings. □

Lemma 1.6. *If $\mathcal{P}$ is irreducible, $|\mathcal{P}| \le 10$, and $\mathcal{P}$ has four or more sh-sticks on its boundary levels, then $\mathcal{P}$ is trivial.*

Proof. Using the previous lemmas, if $|\mathcal{P}|_w \ne 4$, then $\mathcal{P}$ is trivial. Otherwise, $\mathcal{P}$ has at most 10 sh-sticks, four of which are w-sticks (since there are exactly four w-levels) and at least two of which are on intermediate w-levels, $\mathcal{P}$ can have at most four sticks on its boundary w-levels. So, assuming $\mathcal{P}$ has exactly four sticks on its boundary w-levels, it may have three sticks on one boundary w-level and one on the other, or it may have two sticks on both boundary w-levels. The first case, though, can be easily excluded: If $\mathcal{P}$ has three sticks in a single boundary w-level, those three can be reduced to two without changing the knot type of $\mathcal{P}$. To see this, let γ be a three-stick arc on a boundary level with endpoints A and B. Notice that A and B are on opposite corners of a parallelogram R formed from x- and y-sticks. Let γ' be the arc from A to B formed from the x-stick side of R having A as an endpoint and the y-stick side of R having B has an endpoint. Then we may replace γ with γ' in $\mathcal{P}$ without changing the knot type. In a similar way, the arc γ' can be formed from x- and z-sticks or y- and z-sticks.

Thus, we have reduced the problem to showing that if $\mathcal{P}$ has two sh-sticks on each of its two boundary w-levels, then $\mathcal{P}$ is trivial. To this end, we first note that if $\mathcal{P}$ does not contain at least one x-stick, one y-stick and one z-stick, then it must be trivial. To see this, suppose that $\mathcal{P}$ contains no z-sticks. In this case, $\mathcal{P}$ may be "sheered" parallel to the x-direction so that the angles between x-sticks and y-sticks become $\pi/2$. The resulting knot will clearly be of the same type as $\mathcal{P}$, but will be a 10-stick polygon in the simple cubic lattice; consequently, $\mathcal{P}$ must be trivial since the minimal stick number in $\mathbb{Z}^3$ is 12. $\mathcal{P}$ may be similarly sheered if it has no x-sticks or no y-sticks.

Now, if $\mathcal{P}$ has two sh-sticks on each of its two boundary w-levels, since there are exactly four w-sticks in $\mathcal{P}$, there must be exactly one stick on each of the two intermediate w-levels; without loss of generality, let us assume that neither of these two sticks is a z-stick. If we focus our attention to the top boundary w-level, and call the arc in this level γ and its endpoints A and B (as before), we may replace γ with an arc γ' in the same manner as was shown previously so that γ contains no z-sticks. The same procedure can be performed on the bottom boundary w-level as well to create an sh-polygon equivalent to $\mathcal{P}$ that contains no z-sticks on any of its w-levels. By our previous observation involving sheering, we can now see that $\mathcal{P}$ is trivial. □

Figure 1 shows an 11 stick trefoil in the simple hexagonal lattice. We would like to show that there are no nontrivial polygons consisting of 10 or fewer sticks. From the above lemmas, this problem is reduced to checking that every polygon $\mathcal{P}$

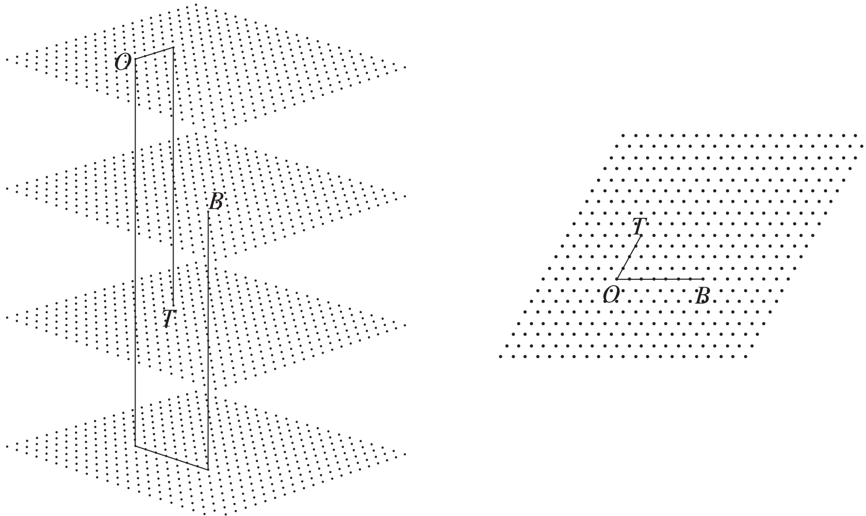

Fig. 7. At left: A partially formed lattice polygon $\mathcal{P}$ with two boundary sticks. At right: The projection of $\mathcal{P}$ onto a w-level.

that is properly leveled with respect to w with $|\mathcal{P}| \leq 10$ and $|\mathcal{P}|_w = 4$ is trivial. By Lemmas 1.2 and 1.3, such a polygon $\mathcal{P}$ must have one w-stick of length 4, two of length 2 and one of length 1, arranged as depicted at left in Fig. 7. By Lemma 1.6 we need to only consider polygons whose boundary w-levels contain, in total, two or three sticks. We may assume without loss of generality that the top level contains one x-stick and the bottom level contains either one or two sticks. To show that any such polygon $\mathcal{P}$ is trivial, we will show that $\mathcal{P}$ is equivalent to one of finitely many polygons, all of which are verified to be trivial.

Roughly speaking, the remainder of the paper is organized as follows.

(1) The relative lengths and arrangements of the two or three sticks on the boundary w-levels of any polygon are enumerated and we argue that there are only finitely many such arrangements that one must consider (see Tables 1 and 2). Each case will result in a partially completed polygon (for example Fig. 7).
(2) The remaining sticks include the fourth w-stick whose endpoints lie in w-levels 2 and 3, and the sticks lying entirely in w-levels 2 and 3. In Lemma 1.8 we bound, in the projection, the distance of the endpoints of these sticks from the endpoints of the w-sticks placed in Step 1.
(3) The exhaustive enumeration of polygons is performed by a computerized algorithm. Each polygon is then verified to be trivial by a reduction algorithm.

The remainder of this section is devoted to making this description rigorous. It will be helpful to first consider a specific example. Suppose that a specific arrangement of boundary sticks has been chosen as in Fig. 7. At right in that figure is a projection of $\mathcal{P}$ onto a w-level in which the w-sticks of $\mathcal{P}$ have been collapsed to points at O, B and T. As described above, we now make a number of choices to complete $\mathcal{P}$. First we will choose the placement of the fourth w-stick. Call its projection A.

With all the w-sticks in place we will choose sticks to connect T with A on w-level 2 and sticks to connect B with A on w-level 3. These sticks must avoid the w-sticks at O, B and T. Notice after w-sticks and boundary sticks are chosen, we can use at most four sticks (total) on w-levels 2 and 3 to create a closed polygon with 10 or fewer sticks. A particular choice of such a completion is shown in Fig. 8.

In the projection, for a given point $P = (a, b)$ of the lattice (the point $a(1, 0) + b(1/2, \sqrt{3}/2)$), consider the three lines passing through this point in the lattice given by the equations $x = a$, $y = b$ and $x + y = a + b$. We will refer to these as the *x-starline* of P, *y-starline* of P and *$x + y$-starline* of P, respectively. The collection of all three lines will be called the *star* of P, denoted $s(P)$. Two points $P = (a, b)$ and $Q = (c, d)$ can be connected via one stick if and only if Q is on $s(P)$. Points P and Q are connected via two sticks by choosing a point in $s(P) \cap s(Q)$. There are at most six intersection points allowing up to six distinct two stick paths connecting P and Q (see Fig. 9). In order to connect P and Q with a three stick path, the first stick will lie on a star of P. Call the endpoints of this stick P and R. Now choose a two stick path connecting R and Q as discussed above.

We can now explain our method for enumerating the cases. We assume the point O is the origin, and after a possible rotation that T is $(a, 0)$. We then divide the plane into a number of regions (using stars of O and T as well as the line $y = a$), as pictured in Fig. 10. We label the corners, line segments and rays on the

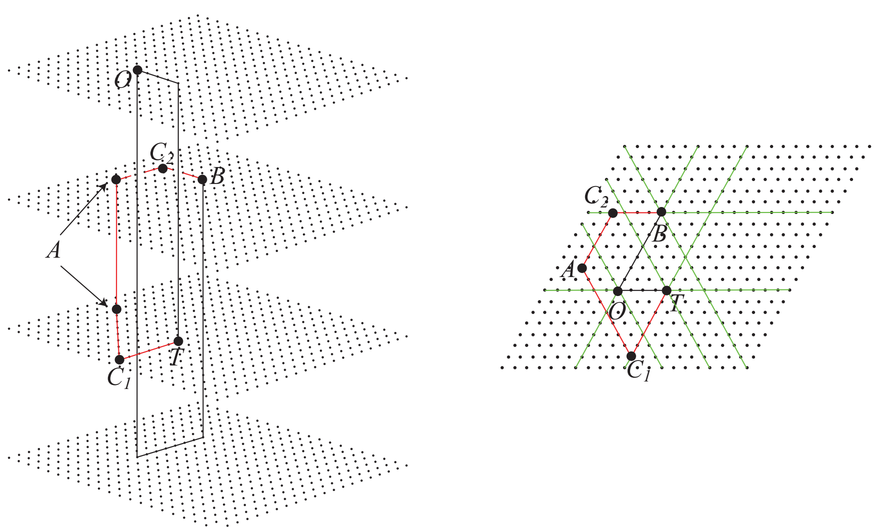

Fig. 8. The completion of $\mathcal{P}$. The green lines in the projection are the stars through O, B and T.

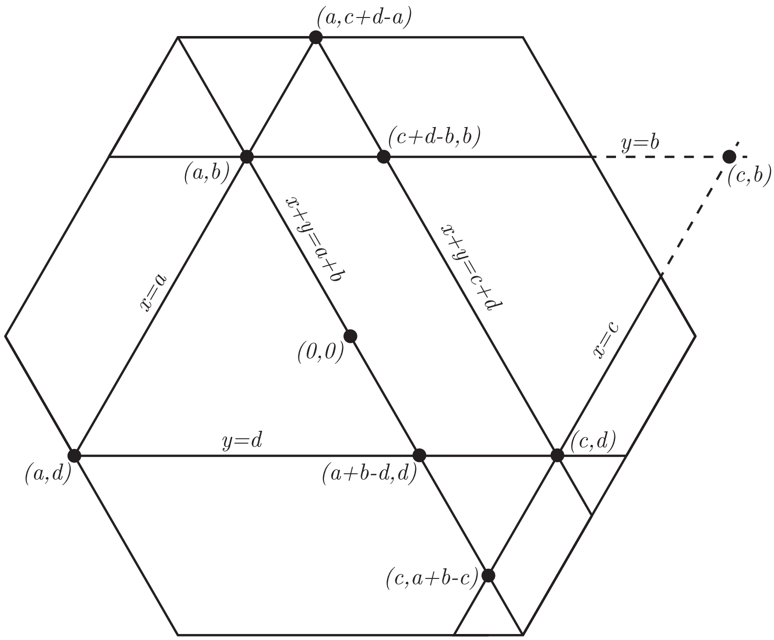

Fig. 9. The intersection points of stars of P and Q.

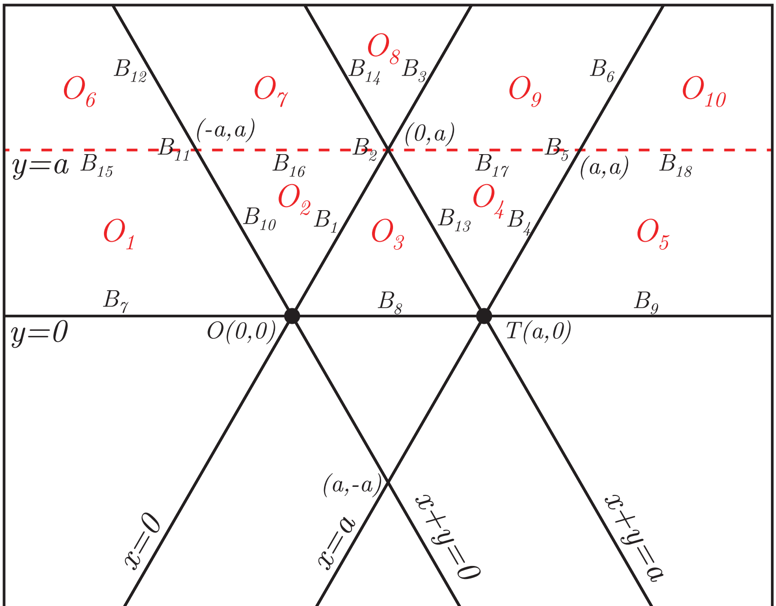

Fig. 10. Partitioning the w-plane into regions.

boundaries of the regions $B_1, \ldots, B_{18}$ and open regions $O_1, \ldots, O_{10}$. Two partitions of the w-plane into regions are considered equivalent if there is a bijection from the set of regions in the first to the set of regions in the second that preserves adjacency of regions. Note that $a \geq 3$ will ensure that $O_1, \ldots, O_{10}$ are nonempty.

Next, consider the placement of B. We will assume without loss of generality that B is placed in the upper-half plane. Once the point B is placed, the stars of O, T and B will partition the w-plane into regions. As one can see from examining Fig. 10, the resulting partitions are equivalent for any two choices of B within a given labeled region (O_i or B_i) and changes as B passes from one region to the next. To keep track of B we will use the scheme depicted in Fig. 11. Thus we need to specify θ, γ, b and c. One can write equations with restrictions defining the boundary regions B_i. For example, B_{15} is defined by $y = a$ with restriction $x < -a$. If B is placed in B_{15}, then $\theta = \pi/3$, $\gamma = \pi$, $b = a$ and $c > a$. Table 1 contains θ, γ, as well as restrictions for b and c for B chosen in B_i. Similarly, Table 2 gives information for the open regions O_i in the partition.

Many choices for B and T will yield equivalent partitions of the w-plane. We want to choose one representative from each class. In choosing a representative, we must also be careful that the boundaries in any region created by the stars of O, T and B are large enough that the interior of the region contains points of the lattice. This will be true provided the length of a boundary segment for a region is at least 3. This will be ensured if $a \geq 3$ and the gaps for strict inequalities in Tables 1 and 2 are at least 3. The last three columns in both of the tables give one representative for each class satisfying these conditions.

We now choose up to three points C_1, C_2, C_3 (called completion points) to complete the polygon. One of these completion points will represent the fourth w-stick. The other completion points will represent endpoints of the other sticks used to complete the arcs on w-levels two and three. Specifically, for completion points C_1, C_2, C_3, we will use the sticks $\overline{TC_1}, \overline{C_1C_2}, \overline{C_2C_3}, \overline{C_3B}$ to complete the arcs.

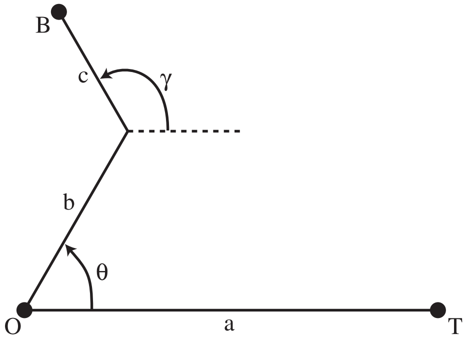

Fig. 11. The arrangement of sticks in the boundary w-levels.

Table 1. The boundary cases.

Region containing B	θ	γ	Restriction for b	Restriction for c	a	b	c
B_1	$\pi/3$	—	$0 < b < a$	—	6	3	—
B_2	$\pi/3$	—	$b = a$	—	3	3	—
B_3	$\pi/3$	—	$b > a$	—	3	6	—
B_4	$\pi/3$	0	$0 < b < a$	$c = a$	6	3	6
B_5	$\pi/3$	0	$b = a$	$c = a$	3	3	3
B_6	$\pi/3$	0	$b > a$	$c = a$	3	6	3
B_7	π	—	$b > 0$	—	3	3	—
B_8	0	—	$0 < b < a$	—	6	3	—
B_9	0	—	$b > a$	—	3	6	—
B_{10}	$2\pi/3$	—	$0 < b < a$	—	6	3	—
B_{11}	$2\pi/3$	—	$b = a$	—	3	3	—
B_{12}	$2\pi/3$	—	$b > a$	—	3	6	—
B_{13}	$\pi/3$	0	$0 < b < a$	$a - b = c$	6	3	3
B_{14}	$\pi/3$	π	$b > a$	$b - a = c$	3	6	3
B_{15}	$\pi/3$	π	$b = a$	$c > a$	3	3	6
B_{16}	$\pi/3$	π	$b = a$	$0 < c < a$	6	6	3
B_{17}	$\pi/3$	0	$b = a$	$0 < c < a$	6	6	3
B_{18}	$\pi/3$	0	$b = a$	$c > a$	3	3	6

Table 2. The interior cases.

Region containing B	θ	γ	Restriction for b	Restriction for c	a	b	c
O_1	$\pi/3$	π	$0 < b < a$	$c > b$	6	3	6
O_2	$\pi/3$	π	$0 < b < a$	$c < b$	9	6	3
O_3	$\pi/3$	0	$0 < b < a$	$0 < c < a - b$	9	3	3
O_4	$\pi/3$	0	$0 < b < a$	$a - b < c < a$	9	6	6
O_5	$\pi/3$	0	$0 < b < a$	$c > a$	6	3	9
O_6	$\pi/3$	π	$b > a$	$c > b$	3	6	9
O_7	$\pi/3$	π	$b > a$	$b - a < c < b$	6	9	6
O_8	$\pi/3$	π	$b > a$	$0 < c < b - a$	3	9	3
O_9	$\pi/3$	0	$b > a$	$0 < c < a$	6	9	3
O_{10}	$\pi/3$	0	$b > a$	$c > a$	3	6	6

From the above tables we see that there are 28 cases to consider concerning the placement of the boundary paths in a 10-stick polygon.

We now argue that once O, B and T are placed, there is a bound on how far away the fourth w-stick needs to be placed from O.

For a point $P = (a, b)$ in the hexagonal lattice, we define $|P|$ to be the distance from P to the origin. Using the distance formula from [2], we have $|P| = \max\{|a|, |b|, |a+b|\}$. For a set of points A, define $\mathrm{maxdist}(A) = \max\{|P| : P \in A\}$.

Lemma 1.7. *If P and Q in the hexagonal lattice are contained within the hexagon of radius r centered at the origin, then $s(P) \cap s(Q)$ is contained in the hexagon of radius $2r$.*

Proof. Assume $P = (a, b)$ and $Q = (c, d)$ lie within a hexagon of radius r centered at the origin. Then $\max\{|a|, |b|, |a + b|\} \leq r$ and $\max\{|c|, |d|, |c + d|\} \leq r$. Note that $s(P) \cap s(Q) = \{(a, c+d-a), (c+d-b, b), (c, b), (a, d), (a+b-d, d)$, and $(c, a+b-c)\}$ (see Fig. 9). One can quickly check that points (x, y) in $s(P) \cap s(Q)$ satisfy $\max\{|x|, |y|, |x+y|\} \leq 2r$. $\square$

Lemma 1.8. *Given a polygon $\mathcal{P}$ containing three or fewer sticks on each w-level with w-stick projections O, T, B (as described previously), there exists an equivalent polygon $\mathcal{P}'$ with w-stick projections O', T', B' satisfying*:

(1) *T' and B' are chosen from the last three columns in Tables* 1 *and* 2
(2) *$\mathcal{P}'$ contains three or fewer sticks on each w-level*
(3) *$\mathcal{P}'$ is contained in a cylinder of radius $r = 2(1 + \text{maxdist}\{\{O', T', B'\} \cup (s(O') \cap s(T')) \cup (s(O') \cap s(B')) \cup (s(T') \cap s(B'))\}) \leq 32$.*

Proof. Up to three completion points exist in $\pi(\mathcal{P})$. One of these completion points will represent the fourth w-stick. The other completion points will represent other endpoints of sticks used to complete the arcs on w-levels two and three. Specifically, for completion points C_1, C_2, C_3, we will use the sticks $\overline{TC_1}, \overline{C_1C_2}, \overline{C_2C_3}, \overline{C_3B}$ to complete the arcs. We will have the first completion point on $s(T)$ and the last completion point on $s(B)$. Note that T and B will satisfy one row of restrictions in Tables 1 and 2. Choose T' and B' to be the one representative for that case given in the last three columns of the tables.

Case 1. Suppose exactly one completion point C_1 exists in $\pi(\mathcal{P})$. Then $C_1 \in s(T) \cap s(B)$. Choose C_1' to be the corresponding intersection point in $s(T') \cap s(B')$.

Case 2. Suppose exactly two completion points C_1 and C_2 exist in $\pi(\mathcal{P})$. The point C_1 is either: (i) a corner of a boundary segment in the partition of O, T and B or (ii) an interior point of a boundary segment in the partition of O, T and B. If (i), then choose C_1' to be the corresponding corner of a boundary segment in the partition of O', T' and B'. If (ii), then choose C_1' to be the first interior point of the corresponding boundary segment in the partition in the direction of $\overrightarrow{TC_1}$. Choose $C_2' \in s(C_1') \cap s(B')$ corresponding to the intersection point $C_2 \in s(C_1) \cap s(B)$.

Case 3. Suppose three completion points C_1, C_2, C_3 exist in $\pi(\mathcal{P})$. The point C_1 is either: (i) a corner of a boundary segment in the partition of O, T and B or (ii) an interior point of a boundary segment in the partition of O, T and B. If (i), then choose C_1' to be the corresponding corner of a boundary segment. If (ii), then choose C_1' to be the first interior point of the corresponding boundary segment in the direction $\overrightarrow{TC_1}$. The point C_3 is either: (i) a corner of a boundary segment in the partition of O, T and B or (ii) an interior point of a boundary segment in the partition of O, T and B. If (i), then choose C_3' to be the corresponding corner of a boundary segment. If (ii), then choose C_3' to be the first interior point of the corresponding boundary segment in the direction $\overrightarrow{BC_3}$. Choose $C_2' \in s(C_1') \cap s(C_3')$ corresponding to the intersection point $C_2 \in s(C_1) \cap s(C_3)$. See Fig. 12 for an example.

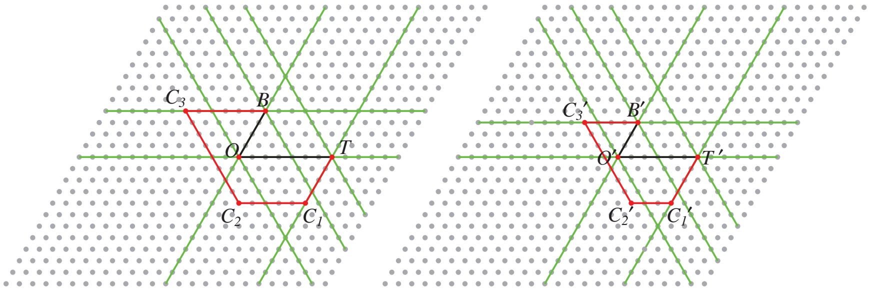

Fig. 12. The projections of the original lattice polygon $\mathcal{P}$ and the corresponding polygon $\mathcal{P}'$ illustrating Case 3 of Lemma 1.8 are pictured.

At this point $\pi(\mathcal{P}')$ is complete. To create the polygon $\mathcal{P}'$ from $\pi(\mathcal{P}')$, place C_i' on the same w-level(s) used for C_i as well as placing $\overline{T'C_1'}, \overline{C_1'C_2'}, \overline{C_2'C_3'}, \overline{C_3'B'}$ on the same w-levels used for $\overline{TC_1}, \overline{C_1C_2}, \overline{C_2C_3}, \overline{C_3B}$. Note the $\mathcal{P}'$ is a well-formed polygon which is equivalent to $\mathcal{P}$ satisfying (1) and (2). By Lemma 1.7, in the worst case scenario (Case 3), C_1', C_2', C_3' would be contained in a hexagon of radius $2(1 + \text{maxdist}\{\{O', T', B'\} \cup (s(O') \cap s(T')) \cup (s(O') \cap s(B')) \cup (s(T') \cap s(B'))\}$. Hence the polygon $\mathcal{P}'$ will satisfy (3). One can verify that $r \leq 32$ over all choices for B' and T'. $\square$

Theorem 1.9. *The stick number of the simple hexagonal lattice is* 11. *Moreover,* $S_{\text{sh}}(3_1) = 11$.

Proof. That the stick number of sh is bounded by 11 is guaranteed by the 11-stick 3_1 sh-knot shown in Fig. 1. Thus $S_{\text{sh}}(3_1) \leq 11$ as well. In order to prove the result, we will show that any sh-polygon $\mathcal{P}$ with 10 or fewer sticks is the trivial polygon. From the lemmas we may assume $\mathcal{P}$ is properly leveled with respect to w, $|\mathcal{P}|_w = 4$, $\mathcal{P}$ uses one stick on w-level 4 whose endpoints are O and T, and $\mathcal{P}$ uses one or two sticks on w-level 1. In addition, using Lemma 1.8 we may assume that O is the origin and B and T come from one of the 28 cases in Tables 1 and 2 and A lies in the cylinder of radius 32. The algorithm for checking all the representatives of all possible 10-stick sh-polygons with above conditions for knottedness proceeds as follows:

(1) For each arrangement of sticks on the boundary w-levels from Tables 1 and 2, construct the stars of O, T and B.
(2) Choose a point A in the cylinder of radius 32.
(3) If it is an arrangement of three sticks on the boundary w-levels (one on top and two on bottom),
    (a) Choose a point C_1 from the star of T and a point C_2 from the star of B. These choices must be made systematically so that the choices of C_1 and C_2 represent all allowable boundaries of regions and corners of regions bounded

by the stars of T and B that may contain a corner of a representative polygon.
(b) Connect C_1 to C_2 with one stick if possible (it is only possible if C_2 is on the star of C_1).
(c) Complete the polygon by adding vertical sticks (as at left in Fig. 8).
(d) Check the resulting polygon for knottedness.

(4) If it is an arrangement of two sticks on the boundary w-levels,
(a) Choose a point C_1 from the star of T and a point C_3 from the star of B. These choices must be made systematically so that the choices of C_1 and C_3 represent all allowable boundaries of regions and corners of regions bounded by the stars of T and B that may contain a corner of a representative polygon.
(b) Choose a point C_2 from each of the remaining allowable regions or boundaries of regions. C_2 must be chosen from the star of C_1 if there are to be two sticks on the second w-level (from the top), or C_3 must be chosen from the star of C_2 if there are to be two sticks on the third w-level.
(c) Connect C_1 to C_2 to C_3.
(d) Check the resulting polygon for knottedness.

The computer algorithm employed to check for knottedness used the procedure described in [3]. The program revealed that every representative was the unknot, which concludes the proof. The C++ code for this program along with the output it generated are available at http://www.math.uttyler.edu/cmann/stick-knots/checked.txt and http://www.math.uttyler.edu/cmann/stick-knots/main.cpp. □

2. Conjectures for Other Lattices

2.1. *The face-centered cubic lattice*

The face-centered cubic lattice has basis

$$\mathcal{F} = \left\{ \langle 1,0,0\rangle, \langle 0,1,0\rangle, \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right\rangle \right\}.$$

We make the following conjecture.

Conjecture 2.1. *The stick number of the face-centered cubic lattice is* 9. *Moreover,* $S_{\mathrm{fcc}}(3_1) = 9$.

In Fig. 13, we provide an example of a 9-stick fcc-knot.

2.2. *The body-centered cubic lattice*

The body-centered cubic lattice has basis

$$\mathcal{B} = \{\langle 2,0,0\rangle, \langle 0,2,0\rangle, \langle 1,1,1\rangle\}.$$

Based on Fig. 14, we have the following conjecture.

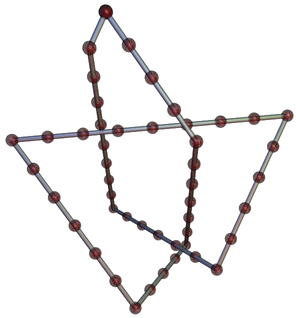

Fig. 13. A 9-stick fcc-knot. The twelve direction vectors pointing from a lattice point to its twelve neighbors are $a_0 = (1,0,0)$, $a_1 = (-1,0,0)$, $a_2 = (0,1,0)$, $a_3 = (0,-1,0)$, $a_4 = (1/2,1/2,1/\sqrt{2})$, $a_5 = (-1/2,-1/2,-1/\sqrt{2})$, $a_6 = (-1/2,1/2,1/\sqrt{2})$, $a_7 = (1/2,-1/2,-1/\sqrt{2})$, $a_8 = (-1/2,-1/2,-1/\sqrt{2})$, $a_9 = (1/2,1/2,-1/\sqrt{2})$, $a_{10} = (1/2,-1/2,1/\sqrt{2})$, $a_{11} = (-1/2,1/2,-1/\sqrt{2})$. Using this notation, the knot depicted above is described by the string $4a_9 6a_2 4a_5 6a_8 8a_0 5a_{11} 6a_6 8a_3 a_{10}$.

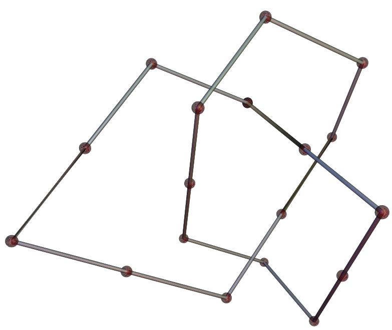

Fig. 14. A 12-stick bcc-knot. The eight direction vectors pointing from a lattice point to its eight neighbors are $a_0 = (1,1,1)$, $a_1 = (-1,-1,-1)$, $a_2 = (-1,1,1)$, $a_3 = (1,-1,-1)$, $a_4 = (-1,-1,1)$, $a_5 = (1,1,-1)$, $a_6 = (1,-1,1)$, $a_7 = (-1,1,-1)$. The string for this knot is then $a_0 a_0 a_3 a_3 a_1 a_4 a_2 a_2 a_7 a_5 a_3 a_6 a_6 a_4 a_4 a_7 a_7 a_5$.

Conjecture 2.2. *The stick number of the body-centered cubic lattice is* 12. *Moreover*, $S_{\mathrm{bcc}}(3_1) = 12$.

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