Discrete Mathematics 340 (2017) 1669–1680

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Unilateral and equitransitive tilings by equilateral triangles[☆]

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ARTICLE INFO

Article history:
Received 6 July 2016
Received in revised form 18 October 2016
Accepted 31 October 2016
Available online 16 December 2016

Keywords:
Tiling
Triangles
Unilateral
Equitransitive

ABSTRACT

A tiling of the plane by polygons is *unilateral* if each edge of the tiling is a side of at most one polygon of the tiling. A tiling is *equitransitive* if for any two congruent tiles in the tiling, there exists a symmetry of the tiling mapping one to the other. It is known that a unilateral and equitransitive (UE) tiling can be made with any finite number of congruence classes of squares. This article addresses the related question, raised in the book *Tilings and Patterns* by Grünbaum and Shephard, of finding all UE tilings by equilateral triangles. In particular, we show that there are only two classes of UE tilings admitted by a finite number of congruence classes of equilateral triangles: one with two sizes of triangles and one with three sizes of triangles.

1. Introduction

A plane *tiling* $\mathscr{T}$ is a countable set of closed topological disks, $\{T_1, T_2, T_3, \ldots\}$, that covers the plane without overlaps (i.e. $\text{int}(T_i) \cap \text{int}(T_j) = \emptyset$ when $i \neq j$). The sets T_i are called the *tiles* of $\mathscr{T}$. Any nontrivial arc along which two tiles of $\mathscr{T}$ meet is called an *edge* of $\mathscr{T}$. Since we will be discussing polygonal tiles in this article, we will be careful to distinguish between the *sides* of the polygons comprising a tiling and the edges of the tiling.

A tiling $\mathscr{T}$ by regular polygons is said to be *equitransitive* if each set of mutually congruent tiles in $\mathscr{T}$ forms one transitivity class with respect to the symmetry group of $\mathscr{T}$. $\mathscr{T}$ is *unilateral* if each edge of $\mathscr{T}$ is a side of at most one polygon (i.e. each edge of $\mathscr{T}$ is a proper subset of a side of one of the two triangle tiles of $\mathscr{T}$ meeting there).

This article aims to settle the question of finding all unilateral and equitransitive (UE) tilings by equilateral triangles of a finite number of sizes. The analogous problem has been previously studied in the case of squares, and it turns out that squares are a rich source of UE tilings. For example, it is known that there exist UE tilings by any number $n \geq 1$ of squares [2, p. 76], and the UE tilings by squares of up to 4 different sizes of squares are completely classified [1–5]. In Fig. 1 we see a UE tiling by squares of 4 sizes. The authors of [2] lament that, except for UE tilings by squares of one or two sizes, such tilings have seen little application, despite their attractiveness and ease of construction.

In [2, p. 73], the authors conjecture that the only equitransitive and unilateral tilings by regular polygons other than squares are two tilings by equilateral triangles (up to the ratios of the sizes of the triangles); one with triangles of two sizes (Fig. 2a) and one with triangles of three sizes (Fig. 2b). Except for squares and triangles, the problem of UE tilings is trivial. In this article, we confirm this conjecture for the class of UE tilings formed from a *finite* number of sizes of equilateral triangles.

☆ This work was supported by the NSF [Grant Number DMS 1460699].

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http://dx.doi.org/10.1016/j.disc.2016.10.022
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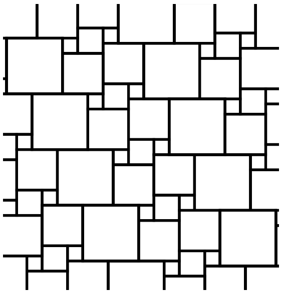


Fig. 1. A UE tiling by squares of four sizes.

(a) EU tiling by equilateral triangles of 2 sizes ($\alpha = 0.5$).

(b) UE tiling by equilateral triangles of 3 sizes ($0 < \alpha < 0.5$).

Fig. 2. EU tiling by equilateral triangles of 2 sizes ($\alpha = 0.5$).

Theorem 1. *For any UE tiling $\mathscr{T}$ by equilateral triangles of finitely many different sizes, let $\alpha(\mathscr{T})$ be the ratio of the side lengths of the smallest triangle of $\mathscr{T}$ to the largest triangle of $\mathscr{T}$.*

1. *$0 < \alpha(\mathscr{T}) \leq 1/2$. Further, if $\alpha(\mathscr{T}) = 1/2$, then there are exactly two sizes of triangles in $\mathscr{T}$, and when $\alpha(\mathscr{T}) < 1/2$ there are exactly 3 sizes of triangles in $\mathscr{T}$.*
2. *For each x with $0 < x \leq 1/2$, there exists exactly one UE tiling by equilateral triangles of finitely many different sizes, $\mathscr{T}_x$, satisfying $\alpha(\mathscr{T}_x) = x$ (up to scale and rigid transformation).*

2. Unextendable configurations

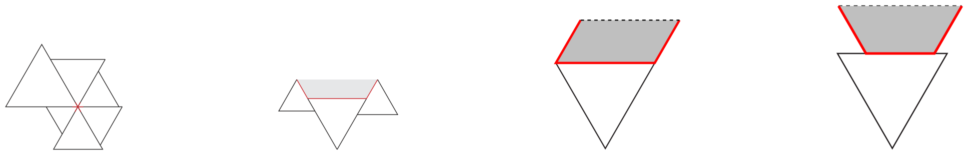
Let $\mathscr{T}$ and α be as defined in Theorem 1. The essence of our proof is to examine the ways in which triangles may share sides with the largest size triangle in the tiling. By a careful process of elimination, we determine that the largest size of triangle must always be surrounded by a pattern of triangles as in Fig. 28b. Once that is established, the equitransitivity property of $\mathscr{T}$ uniquely determines the rest of the tiling.

In establishing that the adjacency pattern around the largest triangle is unique, we first show that some local configurations of triangles may not be extended to a tiling of the plane. In particular, there are four kinds of *unextendable configurations*:

1. A valence-6 vertex (Fig. 3a),
2. a gap where consecutive $2\pi/3$ angles are created on the side of a triangle (a *hex-top* configuration, shaded in Fig. 3b),
3. a gap where a $2\pi/3$ angle is followed by a $\pi/3$ angle on the ends of the base of a single triangle (a *parallelogram* configuration, shaded in Fig. 3c),
4. and, finally, a gap where one of the $2\pi/3$ angles occurs in the middle of the base of the triangle (a *mini hex-top* configuration, shaded in Fig. 3d).

These unextendable configurations are interrelated; their unextendability is due to a geometric version of "proof by infinite descent." For example, we will see that attempting to extend a hex-top configuration always produces a "smaller" hex-top or parallelogram, and attempting to fill these smaller configurations will generate even smaller configurations from among the four types of unextendable configurations, and so on ad infinitum, with the area of spaces to be filled tending to 0. However, such infinite descent is impossible because there is a smallest size triangle!

The crux of the argument presented in this article is that any of these unextendable configurations cannot be present in a UE tiling by a finite number of triangles. Specifically, we will see that a valence-6 vertex leads to a hex-top, which will lead to an even smaller hex-top or parallelogram. The parallelogram space will lead to an even smaller hex-top, parallelogram or



(a) Valence-6 vertex. (b) Hex-top. (c) Parallelogram. (d) Mini hex-top.

Fig. 3. Unextendable configurations.

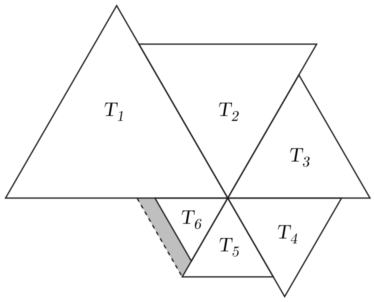


Fig. 4. The unavoidability of a hex-top configuration incident to a valence-6 vertex.

mini hex-top, and a mini hex-top will force an even smaller mini hex-top, an even smaller hex-top, or a parallelogram that is no larger. Thus, the presence of any one of these spaces forces a configuration that is smaller than any available triangle, and so is impossible. Once these unextendable spaces are eliminated as possibilities, it is then straight forward to argue that for a value of α, a unique UE tiling is possible.

Before making this argument, we name one more type of configuration that often arises in our discussion. We say triangle *a overhangs* triangle *b* if triangle *a* and triangle *b* intersect along a side at more than one point and a vertex of triangle *b* lies in the interior of a side of triangle *a*. *Note: Throughout, we will name triangles by their size, so triangle b has size b.*

2.1. Valence-6 vertices

It is easy to show that in a UE tiling a valence-6 vertex forces a hex-top space; later, we will argue that hex-top spaces are unextendable, and so it follows that valence-6 vertices are unextendable. To see that a valence-6 vertex forces a hex-top space, consider a largest triangle T_1 at a valence-6 vertex. Working counterclockwise around this valence-6 vertex, the triangle next to this largest size triangle, T_2, must be strictly smaller than T_1 due to the unilaterality condition. Continuing in a counterclockwise manner, if the next triangle T_3 is larger than T_2, then we form a hex-top space between T_1 and T_3; if T_3 is smaller than T_2, continue to the next triangle T_4, and so on. If we avoid forming hex-top spaces along the way so that $T_1 > T_2 > T_3 > \cdots > T_6$, then we see that a hex-top space is formed between T_1 and T_5, as in Fig. 4.

In the subsequent sections, we show that a hex-top configuration leads to a smaller unextendable configuration and argue why such configurations are unextendable.

2.2. Hex-top configurations

Consider a hex-top space in $\mathscr{T}$ (both as previously defined) and note that since $\mathscr{T}$ is unilateral, at least two triangles must lie adjacent to the base of the hex-top. Thus consider any two neighboring triangles *b* and *c* on the base of a triangle *a*. If the triangle *d* that fills the space between them is smaller than both *b* and *c*, then a smaller hex-top space is created. Otherwise, if $d > b$ or $d > c$ (or both), there is a parallelogram space of size *b* or *c* (Fig. 5). Note that both *b* and *c* are smaller than *a* and so the parallelogram space created will be smaller than the original hex-top space.

2.3. Parallelogram configurations

We consider two cases here:

1. More than two triangles meet the base of the parallelogram configuration.
2. Exactly two triangles meet the base of the parallelogram configuration.

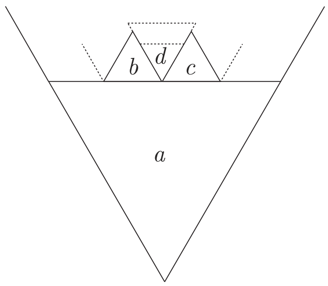


Fig. 5. The only way to fill d is to either create a hex-top space or a parallelogram space, both smaller than a.

(a) $b < c$. (b) $b \geq c$.

Fig. 6. Filling parallelogram configuration with base a.

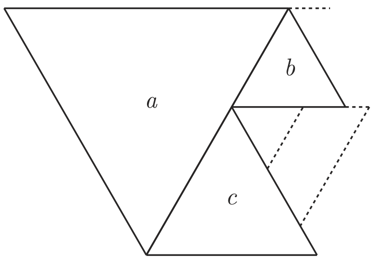


Fig. 7. When $b < c$ in this configuration, a smaller hex-top or parallelogram space is forced to occur.

For Case 1, we first fit a triangle b flush into the obtuse corner formed on base a (Figs. 6a and 6b). Then consider the triangle c that is next to b along the base of a. If $b \leq c$, filling the space between b and c leads to either a smaller hex-top on that triangle or a parallelogram on b (and $b < a$). If $b > c$, filling the space between b and c will either create a small hex-top or a parallelogram on c (since there must be a third triangle along the base a neighboring c). Therefore, if the parallelogram configuration with base a is to be extended, there must be exactly 2 triangles along base a.

For Case 2, there are two subcases to consider when we have two triangles b and c along the base a: $b < c$ and $b \geq c$. In the first case, placing a triangle between b and c that is smaller than b will lead to a smaller hex-top space. However, in placing a triangle larger than b we will create a parallelogram space on b, which is strictly smaller than a (Fig. 7).

Next consider the subcase where $b \geq c$ and focus on the triangle d that fills the space between b and the overhang on a (see Fig. 8). Note that we must have $d < b$, for if $d > b$, that creates a size b parallelogram space on the bottom side of b. If d is smaller than the overhang, we get a hex-top on d. In the case that d extends past the overhang, observe that the rightmost vertex of d (Fig. 9) must have valence 4, and so exactly one of the 3 lines indicated there extends through the vertex in both directions; these possibilities are numbered 1, 2, and 3 in Fig. 9. However, inspection of all 3 possibilities reveals that a smaller unextendable configuration arises at that vertex.

We are left with the case where d is equal to the overhang of triangle e. The triangle f that lies along b and neighboring d must either meet the corner of b or fall short. In the latter case, analysis of the vertex options on f (Fig. 10) shows that this gives rise to a smaller unextendable space. Thus, we know that f must meet flush with the corner of b.

Now consider the options for filling the space above triangle f. First assume that there is more than one triangle along the top of f, so we will have some triangle g that is smaller than f meeting at the vertex shared by d, f and b (note that we must have $g > d$ to avoid creating a hex-top on g). In this case, we look at the vertex options for the top of g (Fig. 11). We know that the vertex must look like option 2, since option 1 creates a parallelogram of size g and option 3 creates a hex top of size g.

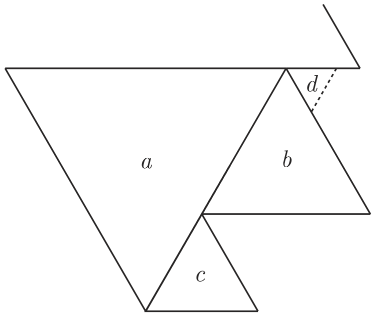


Fig. 8. If d lies short of the overhang, it would be the base of a hex-top.

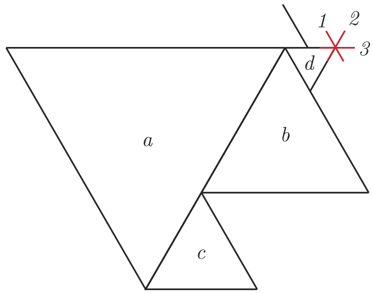


Fig. 9. Vertex option 1 will lead to a smaller parallelogram on d; Vertex option 2 leads to a mini hex-top on part of d; Vertex option 3 causes a hex-top on d.

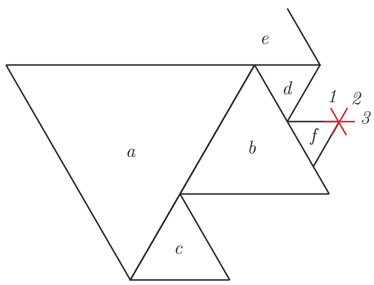


Fig. 10. Vertex option 1 causes a smaller parallelogram space on f; Vertex option 2 also creates a parallelogram on f; Vertex option 3 will make a hex-top on f.

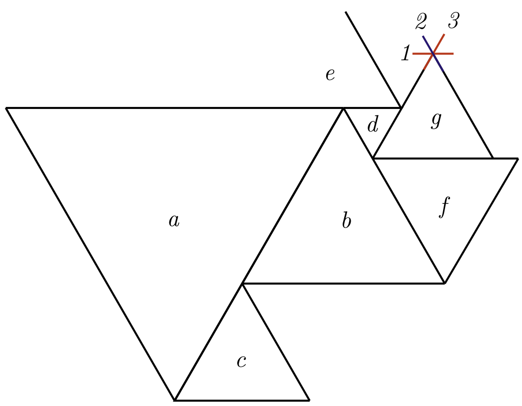


Fig. 11. Vertex option 1 will cause a smaller parallelogram on g, and vertex option 3 creates a hex-top on g. The only possible vertex is type 2.

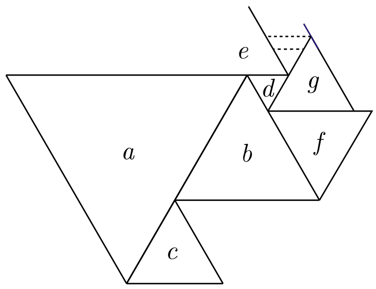


Fig. 12. If e extends higher than g, a smaller hex-top or parallelogram arises.

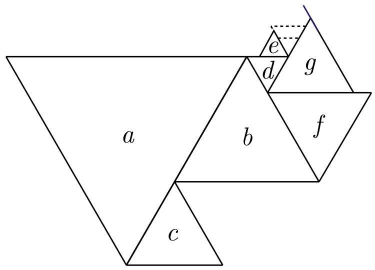


Fig. 13. We are forced to create a smaller unextendable space. Note that e must still be smaller than a since g and d both must be smaller than a.

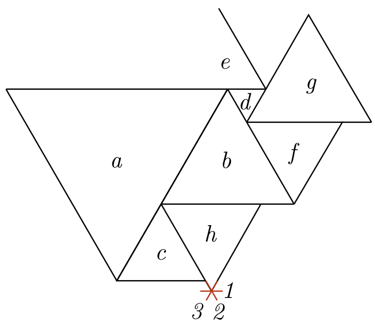


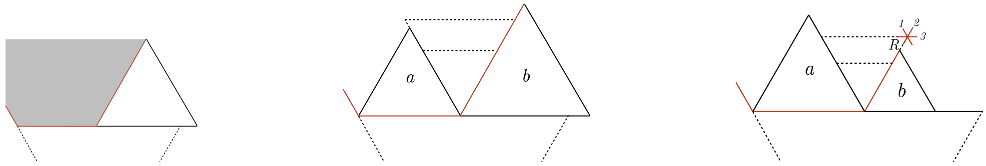
Fig. 14. Vertex option 1 leads to a parallelogram on h; Vertex option 2 causes a hex-top on h; Vertex option 3 creates a mini hex-top on part of h.

In the case that triangle e extends higher than triangle g (Fig. 12), we see that placing a triangle at the vertex shared by d, e, and g forces a smaller hex-top or a smaller parallelogram on this triangle. In the case that triangle g extends higher than triangle e (Fig. 13), placing a triangle at the vertex shared by triangles d, e, and g creates a hex-top on that triangle or a hextop on e (which is smaller than g which is smaller than a). In the case that e and g extend to the same height, a hex-top must be formed on the triangle placed at the vertex shared by triangles d, e, and g.

Since the alternatives lead to unextendable configurations, we know that there can only be one triangle g on the top of f (Fig. 14). And since our tiling is unilateral, we must have that $g > f$. Now consider how we can fill the space between c and b with a triangle h. As we know that g extends past f, we know that $h < b$ or else we will create a parallelogram on f. And by unilaterality, we know that h cannot equal c. Finally, we must have $h > c$ or else it will create a hex-top. Thus we have $c < h < b$, as in Fig. 14. Upon analyzing the vertex options for h, we see that all possibilities lead to an unextendable configuration not bigger than h (which is less than a).

2.4. Mini hex-top configurations

The way that a mini hex-top space differs from a regular hex-top space is that a single triangle may fill the space without violating unilaterality. Note that using more than one triangle to fill a mini hex-top space reduces the problem to filling a normal hex-top space, and so we need consider only placing a single triangle a to complete the base of the mini hex-top space, as in Figs. 15b and 15c.



(a) A mini hex configuration.

(b) $a \leq b$.

(c) $a > b$.

Fig. 15. Filling a mini hex-top configuration

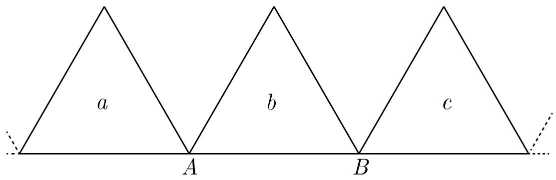


Fig. 16. Lemma 1 states that such a configuration is impossible in a UE tiling by equilateral triangles.

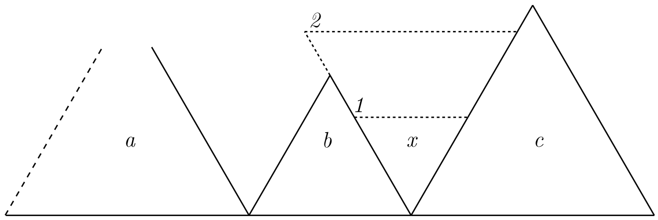


Fig. 17. If $b \leq c$, then x must either be less than b or greater than b. If it is less than b it causes a hex-top, and if it is greater than b it creates a parallelogram on b.

Referring to Fig. 15, there are two possibilities to consider regarding the relationship of a to the triangle b which creates the right portion of the mini hex-top space: $a \leq b$ and $a > b$. When $a \leq b$, placing a triangle between a and b that is smaller than a will result in a smaller hex-top, while placing a triangle larger than a will result in a parallelogram on a. However, in creating a parallelogram on a, we are NOT creating a *smaller* unfillable space, but note that parallelogram configurations are unextendable because they start a cycle of infinite decent. Finally, consider the case $a > b$ (Fig. 15c). Upon trying to fill the space between a and b, placing a triangle smaller than b will lead to a smaller hex-top space. Additionally, placing a triangle larger than a would create a parallelogram on a. Thus, we must place a triangle of size greater than b but less than a. However, we can analyze the possible vertex at the corner R and see that this will lead to smaller unfillable spaces.

3. Consequences of unextendable spaces and proof of Theorem 1

In this section we use the unextendability of the spaces described in the previous section to argue that the largest size triangle in a UE tiling by a finite number of equilateral triangles must be surrounded in a unique way, depending on the value of α. The geometry of the configuration surrounding a largest size triangle forces $0 < \alpha \leq 1/2$, and the uniqueness of the way that a largest triangle is surrounded results in a unique UE tiling for each α.

3.1. No collinear triangles

Lemma 1. *In any UE tiling $\mathscr{T}$ by equilateral triangles, there does not exist a configuration consisting of three triangles a, b, and c whose bases are collinear and such that a neighbors b and b neighbors c, as in Fig. 16.*

Proof. To the contrary, suppose that we have a configuration of three or more triangles lining up and consider two such neighboring triangles, b and c. If $b \leq c$, we see from Fig. 17, placing a triangle x between b and c results in either a hex-top or a parallelogram space.

If $b > c$, we see that placing a triangle x between b and c with $x < c$ will create a hex-top, while $x > b$ will create a parallelogram. Thus we must have $c < x < b$. In this case, the options on x's free vertex will all lead to unfillable spaces (Fig. 18).

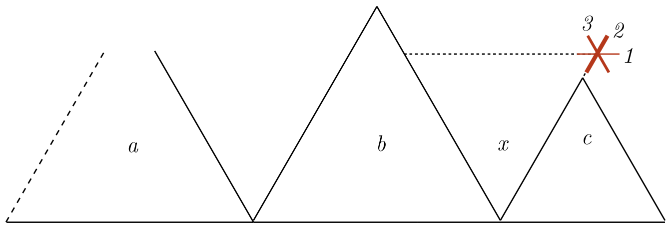


Fig. 18. Vertex options 1, 2 and 3 lead to a mini hex-top, hex-top and parallelogram, respectively.

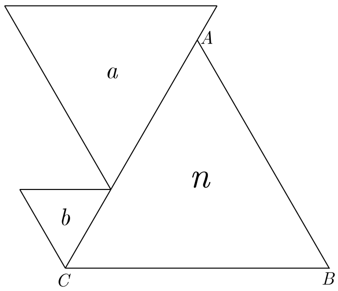


Fig. 19. The side of *a* extends past the corner of *n* creating overhang at vertex *A*.

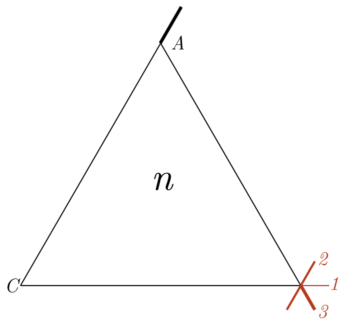


Fig. 20. Vertex option 1 creates a hex-top space, while option 2 creates a parallelogram space.

3.2. No overhang on the largest triangle

Lemma 2. *In a UE tiling by equilateral triangles, the largest size triangle cannot have overhang.*

Proof. For the sake of contradiction, let *n* be the largest size of triangle in a UE tiling having an overhanging triangle along one side, as in Fig. 19 at vertex *A*.

To extend this configuration, one must make a choice that avoids any unfillable spaces at vertex *B*. As shown in Fig. 20, only option 3 does so, creating more overhang.

Next, the same kind of choice must be made at vertex *C*, leaving only option 3 in Fig. 21.

However, even avoiding unfillable spaces until this point still results in a mini-hex-top as demonstrated in Fig. 22. Thus we have reached a contradiction and there cannot be overhang on the largest tile of a UE tiling.

3.3. Adjacents of triangle of size n

As a consequence of Lemmas 1 and 2, we see that exactly two smaller triangles must lie adjacent to each side of the largest size triangle *n*, as in Fig. 23. Note that we have not yet determined the sizes of *n*'s adjacents, so Fig. 23 is not representative of such.

In order to further construct *n*'s corona (a *corona* of a tile is the set of tiles neighboring that tile), we must consider the triangles adjacent to *n*'s adjacents. If we focus on just triangles *a* and *b* in Fig. 23, then the argument we make for these

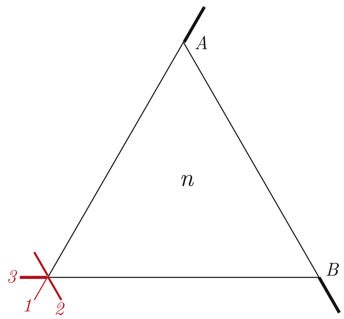


Fig. 21. Again vertex option 1 creates a hex-top space while option 2 creates a parallelogram space.

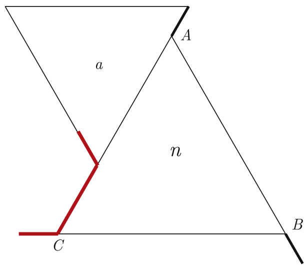


Fig. 22. Since *n* is the largest triangle, some triangle *a* must create the original overhang and fall short of corner *c*. However, as shown in Fig. 21, extending with vertex option 3 creates a mini hex-top, resulting in a contradiction.

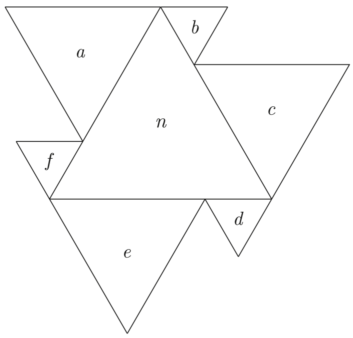


Fig. 23. Triangles adjacent to the largest triangle *n*.

triangles can easily be applied to the remainder of *n*'s adjacents. First notice that there cannot be any overhang on a or b since it would result in a parallelogram space with a or b as the base, as in Fig. 24.

By Lemma 1, the only two other configurations possible are

1. Two tiles, say x and y, lying adjacent against a and b such that $x + y = a + b$.
2. One tile, say m, lies adjacent to a and b such that $m = a + b$.

We now show that the first configuration cannot exist within $\mathscr{T}$. Without loss of generality, suppose $a > b$. Then there are two cases to analyze that involve the relationship of the sizes of x and y to a and b: the first case is when $x > a$ and the second is when $x < a$. In the first case where $x > a$, the tiling process breaks at vertex B since there are only two possible configurations for the vertex environment and both create parallelogram spaces (as demonstrated in Fig. 25a). Similarly, for the case where $x < a$, both possible configurations at vertex A result in a parallelogram space (Fig. 25b).

Therefore, $\mathscr{T}$ must satisfy the second configuration where one tile lies adjacent to a and b such that $m = a + b$. This can be extended to all of n's adjacents and, consequently, n's corona must take on the configuration depicted in Fig. 26.

We now show that tiles m, r, and s in Fig. 26 are all of size n. From Fig. 26, notice that $a + f = b + c = d + e = n$. Without loss of generality assume $a \geq f$ and consider the two possibilities $a = n/2$ or $a > n/2$.

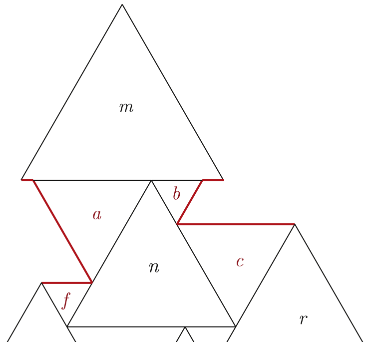
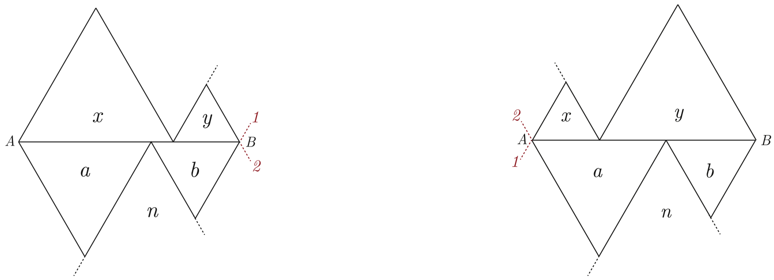


Fig. 24. Overhang on either a or b results in a parallelogram space.



(a) $x > a$.

(b) $x < a$.

Fig. 25. The case $a + b = x + y$ is impossible.

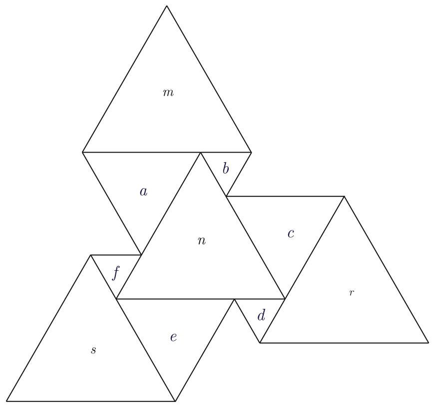


Fig. 26. The established corona for every triangle of the largest size n in a UE tiling by equilateral triangles.

- In light of the fact that $a + b = m \leq n$ (and similarly, $c + d = r \leq n$ and $e + f = s \leq n$), it is easily checked that if $a = n/2$, then $a = b = c = d = e = f = n/2$. That is, if $a = n/2$, then all of the triangles adjacent to n are the same size.
- If $a > n/2$, then $b < n/2$, $c > n/2$, $d < n/2$, $e > n/2$, and $f < n/2$; that is, if one of the triangles adjacent to n is greater than $n/2$, then the sizes of the triangles adjacent to n alternate in size as you read around n.

In both cases $m = r = s = n$. For the first case we demonstrated that if any of a, b, c, d, e, or f is $n/2$, then all are $n/2$. Since $m = a + b$, $r = c + d$, and $s = e + f$, then $m = r = s = n$.

The proof for the second case is slightly more involved. First, assume for the sake of contradiction that at least one of m, r, or s is not of size n. Without loss of generality, suppose $s < n$. Combining the relations $a+f = n$ and $a+b = m \leq n$ shows that $b \leq f$. Similarly, $b + c = n$ and $c + d = r \leq n$ shows $d \leq b$. Continuing around the triangle to the final side, we have

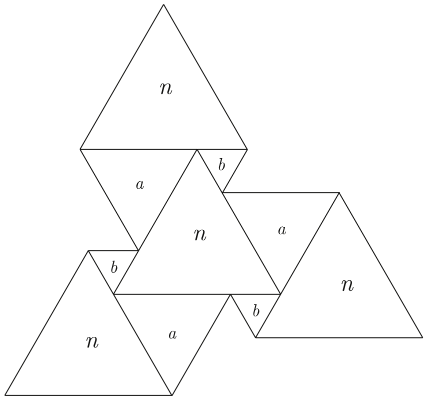


Fig. 27. At most two sizes of triangles make up the tiles adjacent to n, resulting in this pattern.

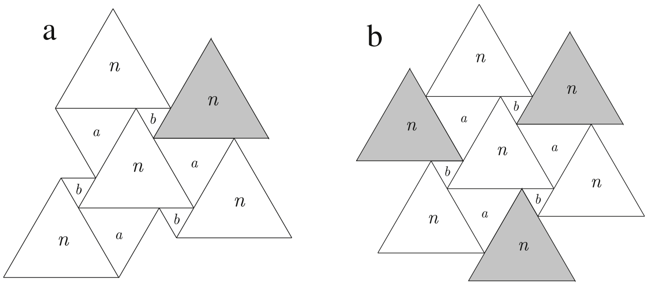


Fig. 28. The corona of n is uniquely determined.

that $d+e=n$ and $e+f=s<n$. This implies that $f<d$, but recall $d\le b\le f$. Since this relationship claims that f is strictly less than itself, we find ourselves at a contradiction. Therefore none of s, m or r can be smaller than n, and so $m=r=s=n$.

And finally, with $m=r=s=n$, it is easily seen that in the case where n's adjacents are not all equivalent, $a=c=e$ and $b=d=f$. This follows from the fact that

$$a+f=a+b=b+c=c+d=d+e=e+f=n.$$

To summarize we observe the following for the triangles that are adjacent to the largest size of triangle in a UE tiling by equilateral triangles:

1. The adjacent triangles are all the same size ($n/2$), or
2. they come in two sizes, a and b, in an alternating pattern, as shown in Fig. 27.

Moreover, notice from Fig. 27 that the corners of a size n triangle must be met by size n triangles.

3.4. There are only two UE tilings by equilateral triangles

The requirement that the size n triangle must be surrounded as in Fig. 27 determines that a unique tiling exists for all possible values of a and b. To see this, first note that in an equitransitive tiling, any two congruent tiles must have congruent coronas. Notice that in Fig. 27, the outer size n triangles each have a neighboring size n triangle (the center size n triangle) meeting them in the middle of an edge. Therefore, at least one size n triangle must meet the middle of a side of the center size n triangle of Fig. 27, giving rise to a configuration as in Fig. 28a. This determines that the size a and size b triangles are surrounded by size n triangles on all sides, and so the full corona of a size n triangle is completely and uniquely determined, as shown in Fig. 28b.

Now that the full corona of the size n triangle is uniquely determined, we see that the complete tiling is uniquely determined as well. Therefore, there are essentially two UE tilings by a finite number of equilateral triangles: one when $a=b=n/2$, and one when $a\ne b$ (Fig. 2), corresponding respectively to the values of α stated in Theorem 1.

We leave the reader with the following open problem: *Are there any UE tilings by equilateral triangles when there is no largest or smallest triangle?*

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